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6 Post-Minkowskian theory: Formulation

In this chapter we embark on a general program to specialize the formulation of general relativity to a description of weak gravitational fields. We will go from the exact theory, which governs the behavior of arbitrarily strong fields, such as those of neutron stars and black holes, to a useful approximation that applies to weak fields, such as those of planets, main-sequence stars, and white dwarfs. This approximation will reproduce the predictions of Newtonian theory, but we will formulate a method that can be pushed systematically to higher and higher order to produce an increasingly accurate description of a weak gravitational field. We shall find that the method is so successful that it can actually handle fields that are not so weak. For example, it provides a perfectly adequate description of gravity at a safe distance from a neutron star, and it can be used as a foundation to study the motion of a binary black-hole system, provided that the mutual gravity between bodies is weak.

The foundation for these methods is "post-Minkowskian theory," the topic of this chapter and the next. In post-Minkowskian theory the strength of the gravitational field is measured by the gravitational constant G, and the Einstein field equations are formally expanded in powers of G. At zeroth post-Minkowskian order there is no field, and one deals with Minkowski spacetime. At first post-Minkowskian order the gravitational field appears as a correction of order G to the Minkowski metric, and the (linearized) field equations are integrated to obtain this correction. The correction is refined by terms of order G^2 in the second post-Minkowskian approximation, and the process is continued until the desired degree of accuracy is achieved.

The formulation of the Einstein field equations that is best suited to this post-Minkowskian expansion was put forward by Landau and Lifshitz, and this framework is introduced in Sec. 6.1. In Sec. 6.2 we refine the Landau–Lifshitz formulation by imposing the harmonic coordinate conditions, and we show that the exact field equations can be expressed as a set of ten wave equations in Minkowski spacetime, with complicated and highly non-linear source terms. We explain how the metric can be systematically expanded in powers of the gravitational constant G and inserted within the wave equations; these are iterated a number of times, and each iteration increases the accuracy of the solution by one power of G.

In Sec. 6.3 we develop mathematical techniques to integrate the wave equation in flat spacetime. We begin by introducing the retarded Green's function for the wave equation, and we explain how the solution can be expressed as an integral over the past light cone of the spacetime point at which it is evaluated. Our methods involve a partition of three-dimensional space into near-zone and wave-zone regions, and we describe how the light-cone integral, decomposed into near-zone and wave-zone contributions, can be evaluated.

In Chapter 7 we implement the techniques developed here and construct the *second post-Minkowskian approximation* to the metric of a weakly curved spacetime. The post-Minkowskian approximation does not rely on an assumption that the matter distribution moves slowly. While this may be the typical context – in a gravitationally bound system, weak gravitational fields induce slow motions – we shall nevertheless divorce the weak-field assumption from a logically distinct slow-motion assumption, which is not required for the developments of this chapter. We shall eventually return to slow motions, however, and formulate an approximation method that incorporates both weak-field and slow-motion aspects. This is the domain of *post-Newtonian theory*, an approximation to general relativity that combines an expansion in powers of G (to measure the strength of the field) with an expansion in powers of c^{-2} (to measure the velocity of the matter distribution). Post-Newtonian theory is informally introduced in Chapter 7, but it is developed more systematically in Chapters 8, 9, and 10. The other main applications of post-Minkowskian theory, gravitational waves and radiation reaction, are the subject of Chapters 11 and 12.

6.1 Landau–Lifshitz formulation of general relativity

6.1.1 New formulation of the field equations

The post-Minkowskian approach to integrate the Einstein field equations is based on the Landau and Lifshitz formulation of these equations. In this framework the main variables are not the components of the metric tensor $g_{\alpha\beta}$ but those of the "gothic inverse metric"

$$\mathfrak{g}^{\alpha\beta} := \sqrt{-g}\, g^{\alpha\beta}, \tag{6.1}$$

where $g^{\alpha\beta}$ is the inverse metric and g the metric determinant. The factor of $\sqrt{-g}$ implies that $\mathfrak{g}^{\alpha\beta}$ is not a tensor; such objects, which differ from tensors by factors of the metric determinant, are known as *tensor densities*. Knowledge of the gothic metric is sufficient to determine the metric itself: note first that $\det[\mathfrak{g}^{\alpha\beta}] = g$, so that g can be directly obtained from the gothic metric; then Eq. (6.1) gives $g^{\alpha\beta}$, which can be inverted to obtain $g_{\alpha\beta}$.

In the Landau–Lifshitz formulation, the left-hand side of the field equations is built from

$$H^{\alpha\mu\beta\nu} := \mathfrak{g}^{\alpha\beta}\mathfrak{g}^{\mu\nu} - \mathfrak{g}^{\alpha\nu}\mathfrak{g}^{\beta\mu}. \tag{6.2}$$

This tensor density is readily seen to possess the same symmetries as the Riemann tensor,

$$H^{\mu\alpha\beta\nu} = -H^{\alpha\mu\beta\nu}, \qquad H^{\alpha\mu\nu\beta} = -H^{\alpha\mu\beta\nu}, \qquad H^{\beta\nu\alpha\mu} = H^{\alpha\mu\beta\nu}. \tag{6.3}$$

It also satisfies the remarkable identity

$$\partial_{\mu\nu} H^{\alpha\mu\beta\nu} = 2(-g)G^{\alpha\beta} + \frac{16\pi G}{c^4}(-g)t^{\alpha\beta}_{\rm LL}, \tag{6.4}$$

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in which $G^{\alpha\beta}$ is the Einstein tensor, and

$$\begin{aligned}(-g)t^{\alpha\beta}_{\rm LL} := \frac{c^4}{16\pi G}\biggl\{ &\partial_\lambda \mathfrak{g}^{\alpha\beta}\partial_\mu \mathfrak{g}^{\lambda\mu} - \partial_\lambda \mathfrak{g}^{\alpha\lambda}\partial_\mu \mathfrak{g}^{\beta\mu} + \frac{1}{2}g^{\alpha\beta}g_{\lambda\mu}\partial_\rho \mathfrak{g}^{\lambda\nu}\partial_\nu \mathfrak{g}^{\mu\rho} \\ &- g^{\alpha\lambda}g_{\mu\nu}\partial_\rho \mathfrak{g}^{\beta\nu}\partial_\lambda \mathfrak{g}^{\mu\rho} - g^{\beta\lambda}g_{\mu\nu}\partial_\rho \mathfrak{g}^{\alpha\nu}\partial_\lambda \mathfrak{g}^{\mu\rho} + g_{\lambda\mu}g^{\nu\rho}\partial_\nu \mathfrak{g}^{\alpha\lambda}\partial_\rho \mathfrak{g}^{\beta\mu} \\ &+ \frac{1}{8}\bigl(2g^{\alpha\lambda}g^{\beta\mu} - g^{\alpha\beta}g^{\lambda\mu}\bigr)\bigl(2g_{\nu\rho}g_{\sigma\tau} - g_{\rho\sigma}g_{\nu\tau}\bigr)\partial_\lambda \mathfrak{g}^{\nu\tau}\partial_\mu \mathfrak{g}^{\rho\sigma}\biggr\} \qquad (6.5)\end{aligned}$$

is the Landau–Lifshitz pseudotensor, so named because it does not transform as a tensor under a general coordinate transformation; the quantity $\partial_{\mu\nu}H^{\alpha\mu\beta\nu}$ is also a pseudotensor, and $(-g)G^{\alpha\beta}$ is a tensor density. Equation (6.4) is valid for any spacetime, whether or not its metric is a solution to the Einstein field equations.

The identity of Eq. (6.4) implies that the Einstein field equations, $G^{\alpha\beta} = (8\pi G/c^4)T^{\alpha\beta}$, can be expressed in the alternative, non-tensorial form

$$\partial_{\mu\nu}H^{\alpha\mu\beta\nu} = \frac{16\pi G}{c^4}(-g)\bigl(T^{\alpha\beta} + t^{\alpha\beta}_{\rm LL}\bigr). \qquad (6.6)$$

As promised, the left-hand side involves $H^{\alpha\mu\beta\nu}$, and the right-hand side is built from $T^{\alpha\beta}$, the energy-momentum tensor of the matter distribution, and $t^{\alpha\beta}_{\rm LL}$. This form of the field equations provides the Landau–Lifshitz pseudotensor with a loose physical interpretation: it represents the distribution of gravitational-field energy in spacetime, which is added to the matter contribution on the right-hand side of the field equations.

By virtue of the antisymmetry of $H^{\alpha\mu\beta\nu}$ in the last pair of indices, we have that the equation

$$\partial_{\beta\mu\nu}H^{\alpha\mu\beta\nu} = 0 \qquad (6.7)$$

holds as an identity. This, together with Eq. (6.6), implies that

$$\partial_\beta\Bigl[(-g)\bigl(T^{\alpha\beta} + t^{\alpha\beta}_{\rm LL}\bigr)\Bigr] = 0. \qquad (6.8)$$

These are conservation equations for the total energy-momentum pseudotensor, expressed in terms of a partial-derivative operator. These equations are equivalent to the usual expression of energy-momentum conservation, $\nabla_\beta T^{\alpha\beta} = 0$, which involves only the matter's energy-momentum tensor and a covariant-derivative operator.

As we have seen, Eqs. (6.6) and (6.8) suggest that $t^{\alpha\beta}_{\rm LL}$ can be interpreted as an energy-momentum (pseudo)tensor for the gravitational field, and this interpretation is supported by the fact that the Landau–Lifshitz pseudotensor is quadratic in $\partial_\mu \mathfrak{g}^{\alpha\beta}$, just as the energy-momentum tensor of the electromagnetic field is quadratic in $\partial_\mu A^\alpha$. This interpretation, however, is not to be taken literally. It is, after all, based on a very specific reformulation of the Einstein field equations, and other reformulations would give rise to other candidates for the energy-momentum pseudotensor. And it is based on a non-tensorial quantity whose numerical value can change arbitrarily by performing a coordinate transformation; indeed, $t^{\alpha\beta}_{\rm LL}$ can be made to *vanish at any selected event in spacetime* by adopting Riemann normal coordinates in the neighborhood of this event (refer to Sec. 5.2.5). The literature abounds

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with attempts to introduce *the* energy-momentum tensor for the gravitational field. Such an object does not exist; do not fall prey to false prophets.

Box 6.1 **Two versions of energy-momentum conservation**

We state in the text that the two versions of energy-momentum conservation, $\nabla_\beta T^{\alpha\beta} = 0$ and $\partial_\beta[(-g)(T^{\alpha\beta} + t^{\alpha\beta}_{\rm LL})] = 0$, are equivalent. In fact, there is an important conceptual difference between these statements. The first equation is a direct consequence of the local conservation of energy-momentum, as observed in a local inertial frame; as such it is valid whether or not Einstein's equations are satisfied, or indeed, whether or not general relativity is the correct theory of gravity. The fact that it is compatible with the Bianchi identity, $\nabla_\beta G^{\alpha\beta} = 0$, is an added feature specific to Einstein's theory. There are alternative theories that lack this consistency, and yet $\nabla_\beta T^{\alpha\beta}$ is still zero.

By contrast, the second conservation equation follows only *after* using Einstein's equations to derive Eq. (6.6). Furthermore, the tedious calculations required to establish that the two versions are equivalent involve inserting the field equations (6.6) at various critical steps along the way.

The bottom line is that the conservation equation $\nabla_\beta T^{\alpha\beta} = 0$ is fundamental; the equation $\partial_\beta[(-g)(T^{\alpha\beta} + t^{\alpha\beta}_{\rm LL})] = 0$ is a consequence of Einstein's equations. If Einstein's equations are satisfied, then either equation may be adopted to express energy-momentum conservation, and the statements are equivalent in this sense.

Equations (6.1)–(6.8) form the core of the Landau–Lifshitz framework. It is out of the question to provide a derivation of these equations (the calculations are straightforward but extremely lengthy), but the following considerations, borrowed from Landau and Lifshitz in their influential book *The classical theory of fields*, will provide at least a partial understanding of where they come from.

Let us write down the Einstein field equations, in their usual tensorial form

$$G^{\alpha\beta} = \frac{8\pi G}{c^4} T^{\alpha\beta}, \tag{6.9}$$

at an event P in spacetime, in a local coordinate system such that $\partial_\gamma g_{\alpha\beta}(P) \stackrel{*}{=} 0$. (We do not demand that $g_{\alpha\beta} \stackrel{*}{=} \eta_{\alpha\beta}$ at P; the special equality sign $\stackrel{*}{=}$ means "equals in the selected coordinate system.") In these coordinates the Riemann tensor at P involves only the metric and its second derivatives, and a short computation reveals that the Einstein tensor is given by

$$\begin{aligned} G^{\alpha\beta} \stackrel{*}{=} \frac{1}{2}\big(& g^{\alpha\lambda}g^{\beta\mu}g^{\nu\rho} + g^{\beta\lambda}g^{\alpha\mu}g^{\nu\rho} - g^{\alpha\lambda}g^{\beta\rho}g^{\mu\nu} - g^{\alpha\mu}g^{\beta\nu}g^{\lambda\rho} \\ & - g^{\alpha\beta}g^{\mu\lambda}g^{\nu\rho} + g^{\alpha\beta}g^{\mu\nu}g^{\lambda\rho}\big)\partial_{\mu\nu}g_{\lambda\rho}. \end{aligned} \tag{6.10}$$

If we now compute $\partial_{\mu\nu}H^{\alpha\mu\beta\nu}$, at the same point P and in the same coordinate system, we find after straightforward manipulations that it is given by

$$\begin{aligned} \partial_{\mu\nu}H^{\alpha\mu\beta\nu} \stackrel{*}{=} (-g)\big(& g^{\alpha\lambda}g^{\beta\mu}g^{\nu\rho} + g^{\beta\lambda}g^{\alpha\mu}g^{\nu\rho} - g^{\alpha\lambda}g^{\beta\rho}g^{\mu\nu} - g^{\alpha\mu}g^{\beta\nu}g^{\lambda\rho} \\ & - g^{\alpha\beta}g^{\mu\lambda}g^{\nu\rho} + g^{\alpha\beta}g^{\mu\nu}g^{\lambda\rho}\big)\partial_{\mu\nu}g_{\lambda\rho}. \end{aligned} \tag{6.11}$$

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To arrive at this result we had to differentiate $(-g)$ using the rule $\partial_\mu(-g) = (-g)g^{\alpha\beta}\partial_\mu g_{\alpha\beta}$, which leads to $\partial_{\mu\nu}(-g) \overset{*}{=} (-g)g^{\alpha\beta}\partial_{\mu\nu}g_{\alpha\beta}$. We also had to relate derivatives of the inverse metric to derivatives of the metric itself; here we have used the rule $\partial_\mu g^{\alpha\beta} = -g^{\alpha\lambda}g^{\beta\rho}\partial_\mu g_{\lambda\rho}$, which leads to $\partial_{\mu\nu}g^{\alpha\beta} \overset{*}{=} -g^{\alpha\lambda}g^{\beta\rho}\partial_{\mu\nu}g_{\lambda\rho}$.

Our results imply that

$$\partial_{\mu\nu}H^{\alpha\mu\beta\nu} \overset{*}{=} 2(-g)G^{\alpha\beta} . \tag{6.12}$$

This is the same as Eq. (6.4), because $(-g)t^{\alpha\beta}_{\rm LL} \overset{*}{=} 0$ at P by virtue of the fact that each term in the Landau–Lifshitz pseudotensor is quadratic in $\partial_\mu \mathfrak{g}^{\alpha\beta}$, which vanishes at P in the selected coordinate system. It is therefore plausible that at any other event in spacetime, and in an arbitrary coordinate system, the identity (6.4) should hold, with a pseudotensor $(-g)t^{\alpha\beta}_{\rm LL}$ that restores all first-derivative terms that were made to vanish at P in the special coordinate system. To show that this pseudotensor takes the specific form of Eq. (6.5) requires a long computation.

6.1.2 Coordinate freedom

The Landau–Lifshitz formulation of general relativity is an *exact reformulation* of the standard form of the theory. No approximations are involved, and no restrictions are placed on the choice of coordinates. It has to be acknowledged, however, that the usefulness of the formalism is largely limited to situations in which (i) the coordinates $x^\alpha = (ct, x^j)$ are modest deformations of the Lorentzian coordinates of flat spacetime, and (ii) $\mathfrak{g}^{\alpha\beta}$ deviates only moderately from the Minkowski metric $\eta^{\alpha\beta}$. For these situations, which form the context of this book, the formalism is an excellent starting point for a systematic approximation method.

In other contexts the Landau–Lifshitz formulation can be a terrible approach. Even a simple problem such as finding the static, spherically symmetric, vacuum solution to the Einstein field equations, the Schwarzschild metric, which took us about six lines of mathematics back in Sec. 5.6, turns out to be a horrible undertaking in the Landau–Lifshitz approach. The lesson is that while the Landau–Lifshitz formulation of the field equations is mathematically equivalent to the tensorial formulation, it is not equivalent when it comes to the ease of performing calculations. In the post-Minkowskian context it is the preferred formulation; in other contexts it decidedly is not.

Given the practical restriction on the coordinate system, it is useful to observe that the Landau–Lifshitz formulation is manifestly invariant under Lorentz transformations, which we express in the general form

$$x^{\mu'} = \Lambda^{\mu'}_{\ \alpha} x^\alpha , \tag{6.13}$$

in which the transformation matrix $\Lambda^{\mu'}_{\ \alpha}$ is constant and possesses a unit determinant. (In fact, the formalism is invariant under all transformations that are linear in the coordinates, so long as the transformation matrix possesses a unit determinant; this ensures that g is not changed during the transformation. The transformation can also be generalized to include uniform translations, $x^\mu \to x^\mu + c^\mu$, where c^μ is a constant vector.) It is easy to show that $\mathfrak{g}^{\alpha\beta}$ and its partial derivatives transform as tensors under this class of transformations,

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and from this observation it follows immediately that all equations of the formalism are invariant under the transformation of Eq. (6.13).

6.1.3 Integral conservation identities

Because they involve a partial-derivative operator, the differential identities of Eq. (6.8) can immediately be turned into integral identities. We consider a three-dimensional region V, a fixed (time-independent) domain of the spatial coordinates x^j, bounded by a two-dimensional surface S. We assume that V contains at least some of the matter (so that $T^{\alpha\beta}$ is non-zero somewhere within V), but that S does not intersect any of the matter (so that $T^{\alpha\beta} = 0$ everywhere on S).

Total momentum and angular momentum: Volume integrals

We formally define a total momentum four-vector $P^\alpha[V]$ associated with the region V by the three-dimensional integral

$$P^\alpha[V] := \frac{1}{c}\int_V (-g)\big(T^{\alpha 0} + t^{\alpha 0}_{\rm LL}\big)\, d^3x. \tag{6.14}$$

This total momentum includes a contribution from the matter's momentum density $c^{-1}T^{\alpha 0}$, and a contribution from the gravitational field represented by $c^{-1}t^{\alpha 0}_{\rm LL}$; the factor of $(-g)$ is inserted so that we can take advantage of the conservation identities of Eq. (6.8). In flat spacetime and in Lorentzian coordinates, $P^\alpha[V]$ would have a firm interpretation as a total momentum vector associated with the energy-momentum tensor $T^{\alpha\beta}$. In curved spacetime, and in a coordinate system that cannot be assumed to be Lorentzian, the quantity defined by Eq. (6.14) does not have any direct physical meaning. It is, nevertheless, a useful quantity to introduce, as we shall have occasion to recognize.

The momentum four-vector can be decomposed into a time component $P^0[V]$ and a spatial three-vector $P^j[V]$. The time component can be used to define an energy $E[V] := cP^0[V]$ associated with the region V. Alternatively, we can define a total mass

$$M[V] := \frac{1}{c^2}\int_V (-g)\big(T^{00} + t^{00}_{\rm LL}\big)\, d^3x. \tag{6.15}$$

The three-momentum is given by

$$P^j[V] := \frac{1}{c}\int_V (-g)\big(T^{j0} + t^{j0}_{\rm LL}\big)\, d^3x. \tag{6.16}$$

In a similar way we introduce a total angular-momentum tensor $J^{\alpha\beta}[V]$ associated with the region V. This is defined by

$$J^{\alpha\beta}[V] := \frac{1}{c}\int_V \Big[x^\alpha(-g)\big(T^{\beta 0} + t^{\beta 0}_{\rm LL}\big) - x^\beta(-g)\big(T^{\alpha 0} + t^{\alpha 0}_{\rm LL}\big)\Big]\, d^3x, \tag{6.17}$$

and we note that the tensor is antisymmetric in its indices. The interpretation of $J^{\alpha\beta}[V]$ is easier to identify once it is decomposed into time and spatial components. The antisymmetry

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of the tensor implies that $J^{00}[V] = 0$. The time-space components can be expressed in the form

$$c^{-1} J^{0j}[V] = P^j[V]t - M[V]R^j[V], \tag{6.18}$$

where

$$R^j[V] := \frac{1}{M[V]c^2} \int_V (-g)\big(T^{00} + t^{00}_{\rm LL}\big) x^j \, d^3x \tag{6.19}$$

represents the position of the center-of-mass of the region V. Equation (6.18) reveals that when $c^{-1}J^{0j}[V]$ is a constant, it fixes the position of the center-of-mass at $t = 0$; when it is not a constant it measures the extent by which the center-of-mass fails to move with a total momentum $P^j[V]$. The spatial components of the angular-momentum tensor are

$$J^{jk}[V] = \frac{1}{c} \int_V \Big[x^j \, (-g)\big(T^{k0} + t^{k0}_{\rm LL}\big) - x^k \, (-g)\big(T^{j0} + t^{j0}_{\rm LL}\big) \Big] \, d^3x, \tag{6.20}$$

and this is best recognized in its equivalent vectorial form

$$J^j[V] := \frac{1}{2} \epsilon^j_{\ pq} J^{pq}[V] = \frac{1}{c} \int_V \epsilon^j_{\ pq} x^p \, (-g)\big(T^{q0} + t^{q0}_{\rm LL}\big) \, d^3x, \tag{6.21}$$

where ϵ_{jpq} is the completely antisymmetric permutation symbol. The integrand is the cross product between the position vector x^p and the momentum density $c^{-1}(-g)(T^{q0} + t^{q0}_{\rm LL})$ within V, and it is natural to interpret the integral as the total angular momentum contained in this region.

Total momentum and angular momentum: Surface integrals

The total momentum $P^\alpha[V]$ and angular momentum $J^{\alpha\beta}[V]$ were defined previously in terms of integrals over the three-dimensional region V. It is possible to provide alternative definitions in terms of surface integrals over the two-dimensional surface S that surrounds this region. This is advantageous when the volume integrals of Eq. (6.14) and (6.17) are ill-defined or difficult to compute.

Substituting Eq. (6.6) into Eq. (6.14) gives

$$P^\alpha[V] = \frac{c^3}{16\pi G} \int_V \partial_{\mu\nu} H^{\alpha\mu 0\nu} \, d^3x.$$

Summation over ν must exclude $\nu = 0$, because $H^{\alpha\mu 00} = 0$. We therefore have

$$P^\alpha[V] = \frac{c^3}{16\pi G} \int_V \partial_k \big(\partial_\mu H^{\alpha\mu 0k}\big) \, d^3x,$$

and this can be written as a surface integral by invoking Gauss's theorem. We have

$$P^\alpha[V] := \frac{c^3}{16\pi G} \oint_S \partial_\mu H^{\alpha\mu 0k} \, dS_k, \tag{6.22}$$

where dS_k is an outward-directed surface element on the two-dimensional surface S. Equation (6.22) can be adopted as an alternative definition for the total momentum enclosed

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by S; $H^{\alpha\mu 0k}$ must then be constructed from a solution to Einstein's equations for the given distribution of matter.

As before the momentum four-vector can be decomposed into time and spatial components. We have that the total mass $M[V]$ can be expressed as

$$M[V] := \frac{c^2}{16\pi G} \oint_S \partial_j H^{0j0k}\, dS_k, \tag{6.23}$$

and the total three-momentum is

$$P^j[V] := \frac{c^3}{16\pi G} \oint_S \partial_n H^{jn0k}\, dS_k - \frac{c^2}{16\pi G}\frac{d}{dt} \oint_S H^{0j0k}\, dS_k. \tag{6.24}$$

With similar manipulations we arrive at a surface-integral definition for the total angular momentum. One of the two terms that occur within the volume integral when we substitute Eq. (6.6) into Eq. (6.17) is $x^\alpha \partial_{k\mu} H^{\beta\mu 0k}$, which can be expressed as $\partial_k(x^\alpha \partial_\mu H^{\beta\mu 0k}) + \partial_\mu H^{\mu\beta 0\alpha}$. The first term gives rise to a surface integral, and the second term can be expanded as $\partial_0 H^{0\beta 0\alpha} + \partial_k H^{k\beta 0\alpha}$; in this, the first term can be ignored because it is symmetric in α and β, and the second term gives rise to another surface integral. Collecting results, we arrive at

$$J^{\alpha\beta}[V] := \frac{c^3}{16\pi G} \oint_S \left(x^\alpha \partial_\mu H^{\beta\mu 0k} - x^\beta \partial_\mu H^{\alpha\mu 0k} + H^{0\alpha k\beta} - H^{0\beta k\alpha}\right) dS_k, \tag{6.25}$$

and this can be adopted as an alternative definition for the total angular momentum enclosed by S.

The decomposition of $J^{\alpha\beta}[V]$ into time and spatial components first returns Eq. (6.18) together with the alternative expression

$$M[V]R^j[V] := \frac{c^2}{16\pi G} \oint_S \left(x^j \partial_n H^{0n0k} - H^{0j0k}\right) dS_k \tag{6.26}$$

for the position of the center-of-mass. It also returns

$$\begin{aligned} J^{jk}[V] :=\ & \frac{c^3}{16\pi G} \oint_S \left(x^j \partial_m H^{km0n} - x^k \partial_m H^{jm0n} + H^{0jnk} - H^{0knj}\right) dS_n \\ & - \frac{c^2}{16\pi G}\frac{d}{dt} \oint_S \left(x^j H^{0k0n} - x^k H^{0j0n}\right) dS_n \end{aligned} \tag{6.27}$$

as an alternative definition for the angular-momentum tensor.

Conservation statements

To obtain the conservation statements satisfied by $P^\alpha[V]$ and $J^{\alpha\beta}[V]$, we differentiate their defining expressions (in terms of volume integrals) with respect to x^0 and use the local conservation identity of Eq. (6.8). Starting with Eq. (6.14), we get

$$\begin{aligned} \frac{d}{dx^0} P^\alpha[V] &= \frac{1}{c} \int_V \partial_0 \Big[(-g)\big(T^{\alpha 0} + t^{\alpha 0}_{\rm LL}\big)\Big]\, d^3x \\ &= -\frac{1}{c} \int_V \partial_k \Big[(-g)\big(T^{\alpha k} + t^{\alpha k}_{\rm LL}\big)\Big]\, d^3x. \end{aligned} \tag{6.28}$$

Converting this to a surface integral, and recalling our previous assumption that S does not intersect the matter distribution, so that $T^{\alpha\beta} = 0$ on S, we arrive at

$$\dot{P}^{\alpha}[V] = -\oint_S (-g) t^{\alpha k}_{\rm LL} \, dS_k, \tag{6.29}$$

in which an overdot indicates differentiation with respect to $t := x^0/c$. The rate of change of $P^{\alpha}[V]$ is therefore expressed as a flux integral over S, and the flux is measured by the Landau–Lifshitz pseudotensor (recall the definitions of fluxes provided back in Sec. 4.2). Equation (6.29) gives rise to the individual statements

$$\dot{M}[V] = -\frac{1}{c}\oint_S (-g) t^{0k}_{\rm LL} \, dS_k \tag{6.30}$$

and

$$\dot{P}^{j}[V] = -\oint_S (-g) t^{jk}_{\rm LL} \, dS_k \tag{6.31}$$

for the fluxes of mass and momentum three-vector across S.

Proceeding along similar lines for the angular-momentum tensor, we arrive at

$$\dot{J}^{\alpha\beta}[V] = -\oint_S \bigl[x^{\alpha}(-g) t^{\beta k}_{\rm LL} - x^{\beta}(-g) t^{\alpha k}_{\rm LL} \bigr] \, dS_k. \tag{6.32}$$

The symmetry of $t^{\alpha\beta}_{\rm LL}$ was essential in obtaining this result. When decomposed into time and spatial components, the statement becomes

$$c^{-1}\dot{J}^{0j}[V] = \dot{P}^{j}[V] t + \frac{1}{c}\oint_S x^{j}(-g) t^{0k}_{\rm LL} \, dS_k \tag{6.33}$$

and

$$\dot{J}^{jk}[V] = -\oint_S \bigl[x^{j}(-g) t^{kn}_{\rm LL} - x^{k}(-g) t^{jn}_{\rm LL} \bigr] \, dS_n. \tag{6.34}$$

Equation (6.33), when combined with Eq. (6.18), implies that

$$\frac{d}{dt}\bigl(M[V] R^{j}[V] \bigr) = P^{j}[V] - \frac{1}{c}\oint_S x^{j}(-g) t^{0k}_{\rm LL} \, dS_k. \tag{6.35}$$

6.1.4 Total mass, momentum, and angular momentum

The limit in which V is taken to include all of three-dimensional space is particularly interesting. In this limit $P^{\alpha}[V]$ is known to coincide with the Arnowitt–Deser–Misner four-momentum of an asymptotically-flat spacetime, and its physical interpretation as a measure of total momentum is robust. This statement is true whenever the coordinates x^{α}

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coincide with a Lorentzian system at infinity; the coordinates do not have to be Lorentzian (and indeed, they could not be) at finite spatial distances.

Recalling the definitions of Eqs. (6.15) and (6.23), we define the total mass of the spacetime as

$$M := \frac{1}{c^2}\int_{\text{all space}} (-g)\big(T^{00} + t^{00}_{\text{LL}}\big)\, d^3x \tag{6.36a}$$

$$:= \frac{c^2}{16\pi G}\oint_\infty \partial_j H^{0j0k}\, dS_k. \tag{6.36b}$$

Recalling the definitions of Eqs. (6.16) and (6.24), we define the total three-momentum of the spacetime as

$$P^j := \frac{1}{c}\int_{\text{all space}} (-g)\big(T^{j0} + t^{j0}_{\text{LL}}\big)\, d^3x \tag{6.37a}$$

$$:= \frac{c^3}{16\pi G}\oint_\infty \partial_n H^{jn0k}\, dS_k - \frac{c^2}{16\pi G}\frac{d}{dt}\oint_\infty H^{0j0k}\, dS_k. \tag{6.37b}$$

Recalling the definitions of Eqs. (6.20) and (6.27), we define the total angular-momentum three-tensor of the spacetime as

$$J^{jk} := \frac{1}{c}\int_{\text{all space}} \Big[x^j(-g)\big(T^{k0} + t^{k0}_{\text{LL}}\big) - x^k(-g)\big(T^{j0} + t^{j0}_{\text{LL}}\big)\Big]\, d^3x \tag{6.38a}$$

$$:= \frac{c^3}{16\pi G}\oint_\infty \big(x^j\partial_m H^{km0n} - x^k\partial_m H^{jm0n} + H^{0jnk} - H^{0knj}\big)\, dS_n$$
$$- \frac{c^2}{16\pi G}\frac{d}{dt}\oint_\infty \big(x^j H^{0k0n} - x^k H^{0j0n}\big)\, dS_n. \tag{6.38b}$$

And finally, recalling Eqs. (6.19) and (6.26), we define

$$R^j := \frac{1}{Mc^2}\int_{\text{all space}} \big(T^{00} + t^{00}_{\text{LL}}\big)x^j\, d^3x \tag{6.39a}$$

$$:= \frac{c^2}{16\pi GM}\oint_\infty \big(x^j\partial_n H^{0n0k} - H^{0j0k}\big)\, dS_k \tag{6.39b}$$

as the position of the center-of-mass for the entire spacetime. The mass, momentum, angular momentum, and center-of-mass position of a spacetime can be defined either in terms of volume integrals over all space, or in terms of surface integrals at infinity. The surface integrals are especially powerful because they allow us to determine these quantities directly from the asymptotic behavior of the metric at large distances; an intimate knowledge of the material source is not required. This is reminiscent of the situation in electrodynamics: the total electric charge can be determined by integrating the normal component of the electric field over a surface enclosing the charge, and nothing need be known of the detailed distribution of charge within the surface.

Equations (6.30), (6.31), and (6.34) imply that the total mass M, total momentum P^j, and total angular momentum J^{jk} are constant in time whenever the surface integrals vanish

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in the limit $S \to \infty$. Under these circumstances, we have the conservation statements

$$M = \text{constant}, \qquad P^j = \text{constant}, \qquad J^{jk} = \text{constant}. \tag{6.40}$$

Furthermore, it can be shown that whenever the surface integrals vanish, the volume integrals of Eqs. (6.14) and (6.17) can be evaluated on any spacelike hypersurface and produce the same result. In particular, the momentum four-vector can be evaluated on a surface of simultaneity $t' = \text{constant}$ that is obtained from the original surface $t = \text{constant}$ by a Lorentz transformation; this observation can be used to show that P^α transforms as a four-vector under the transformation of Eq. (6.13).

In a similar way, Eq. (6.35) implies that $M\dot{R}^j = P^j$ whenever its surface integral vanishes, and whenever M itself is a constant. Assuming that $\boldsymbol{P}$ also is constant, we have

$$M\boldsymbol{R}(t) = M\boldsymbol{R}(0) + \boldsymbol{P}\, t, \tag{6.41}$$

where $\boldsymbol{R}(0)$ is the position of the center-of-mass at $t = 0$. This equation states that the center-of-mass moves uniformly with a velocity $\boldsymbol{P}/M$ (recall that $M = P^0/c$) .

When Eq. (6.40) holds it is natural to adopt a reference frame in which $\boldsymbol{P}$ vanishes. This can always be achieved by performing a Lorentz transformation described by Eq. (6.13) and directing the boost in the direction of the momentum; the boost parameter must be set equal to $v = |\boldsymbol{P}|/M$. Once this is accomplished, it is also natural to place the origin of the spatial coordinates at the center-of-mass $\boldsymbol{R}$. This can always be achieved by translating the coordinates according to $\boldsymbol{x} \to \boldsymbol{x} - \boldsymbol{c}$, with $\boldsymbol{c}$ denoting a constant vector. It is easy to see that the translation changes the position of the center-of-mass according to $\boldsymbol{R} \to \boldsymbol{R} - \boldsymbol{c}$, and choosing $\boldsymbol{c} = \boldsymbol{R}$ places the center-of-mass at the origin of the spatial coordinates.

These choices define the *center-of-mass frame* of the spacetime:

$$\text{center-of-mass frame:} \qquad P^j = 0, \qquad R^j = 0. \tag{6.42}$$

As we have seen, this choice can be made whenever $\boldsymbol{P}$ is a constant vector, and whenever $M\dot{\boldsymbol{R}} = \boldsymbol{P}$. These conditions are fulfilled whenever the surface integrals of Eqs. (6.31) and (6.35) vanish when $S \to \infty$. This always happens when the spacetime is stationary. In the context of a radiating spacetime, however, the surface integrals cannot be assumed to vanish; in fact, the mass, momentum, and angular momentum of the spacetime are typically seen to change with time because the radiation transports energy, momentum, and angular momentum away from the source. Fortunately this effect can often be neglected in the context of approximate calculations.

We conclude this discussion with an illustration: we use the surface integrals to calculate the mass, momentum, and angular momentum of the Schwarzschild spacetime, first encountered back in Sec. 5.6. Expressing the metric of Eq. (5.163) in Cartesian coordinates, we find that

$$g_{00} = -\left(1 - \frac{R}{r}\right), \tag{6.43a}$$

$$g_{jk} = \delta_{jk} + \left(1 - \frac{R}{r}\right)^{-1} \frac{R}{r} n_j n_k , \tag{6.43b}$$

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where $R := 2GM/c^2$ and $n^j := x^j/r$. It is then simple to show that $g = -1$ and

$$\mathfrak{g}^{00} = -\left(1 - \frac{R}{r}\right)^{-1}, \tag{6.44a}$$

$$\mathfrak{g}^{jk} = \delta^{jk} - \frac{R}{r} n^j n^k. \tag{6.44b}$$

We next compute $H^{\alpha\mu 0j}$ by substituting Eqs. (6.44) into Eq. (6.2), and insert the result within Eq. (6.22) to calculate $P^\alpha[r]$, the momentum vector associated with a surface S of constant r. The computations involve the surface element $dS_j = r^2 n_j\, d\Omega$ (where $d\Omega := \sin\theta\, d\theta d\phi$ is an element of solid angle), and they lead to $P^j[r] = 0$ and

$$M[r] = M\frac{r}{r - 2GM/c^2}. \tag{6.45}$$

The spatial momentum vanishes (as expected, since the coordinates are centered on the black hole), and in the limit $r \to \infty$ our previous result reduces to

$$M[\infty] = M. \tag{6.46}$$

The total energy is $cP^0[\infty] = Mc^2$, and M is recognized as the total gravitational mass of the Schwarzschild spacetime. A similar calculation reveals that the center-of-mass is situated at $R^j = 0$ and that the angular momentum vanishes.

6.2 Relaxed Einstein equations

6.2.1 Harmonic coordinates and a wave equation

It is advantageous at this stage to impose the four conditions

$$\partial_\beta \mathfrak{g}^{\alpha\beta} = 0 \tag{6.47}$$

on the gothic inverse metric. These are known as the *harmonic coordinate conditions*, and they were first encountered back in Sec. 5.6, see Eq. (5.177), in the context of the Schwarzschild solution. It is also useful to introduce the potentials

$$h^{\alpha\beta} := \eta^{\alpha\beta} - \mathfrak{g}^{\alpha\beta}, \tag{6.48}$$

where $\eta^{\alpha\beta} := \mathrm{diag}(-1, 1, 1, 1)$ is the Minkowski metric expressed in Lorentzian coordinates $(x^0 := ct, x^j)$. In terms of these potentials the harmonic coordinate conditions read

$$\partial_\beta h^{\alpha\beta} = 0, \tag{6.49}$$

and in this context they are usually referred to as the *harmonic gauge conditions*. We observe that the harmonic conditions are preserved under the Lorentz transformations of

Eq. (6.13), and that the potentials $h^{\alpha\beta}$ transform as a tensor under this restricted class of coordinate transformations.

Box 6.2 Existence of harmonic coordinates

It seems plausible that the four harmonic coordinate conditions of Eq. (6.47) can always be imposed, given the four degrees of coordinate freedom inherent to general relativity, but it is worthwhile to see this explicitly. Given an initial coordinate system in which $\partial_\beta \mathfrak{g}^{\alpha\beta} \neq 0$, we make a coordinate transformation to $x'^\mu = f^\mu(x^\alpha)$. It is then straightforward to show that in the new coordinates,

$$\partial_{\nu'} \mathfrak{g}^{\mu'\nu'} = \sqrt{-g'}\, \Box_g f^\mu(x^\alpha)\,,$$

where $\Box_g := g^{\mu\nu}\nabla_\mu\nabla_\nu$ is the curved spacetime d'Alembertian operator acting on each one of the four functions f^μ, treated as a scalar function of x^α. Choosing each function to be *harmonic*, that is, a solution to $\Box_g f^\mu = 0$, ensures that the harmonic coordinate conditions will hold in the new coordinates.

The introduction of the potentials $h^{\alpha\beta}$ and the imposition of the harmonic gauge conditions simplify the appearance of the Einstein field equations. It is easy to verify that the left-hand side becomes

$$\partial_{\mu\nu} H^{\alpha\mu\beta\nu} = -\Box h^{\alpha\beta} + h^{\mu\nu}\partial_{\mu\nu}h^{\alpha\beta} - \partial_\mu h^{\alpha\nu}\partial_\nu h^{\beta\mu}, \tag{6.50}$$

where $\Box := \eta^{\mu\nu}\partial_{\mu\nu}$ is the flat-spacetime wave operator. The right-hand side of the field equations stays essentially unchanged, but the harmonic conditions do slightly simplify the form of the Landau–Lifshitz pseudotensor; as can be seen from Eq. (6.5), the first two terms of $(-g)t^{\alpha\beta}_{\rm LL}$ vanish in harmonic coordinates. Isolating the wave operator on the left-hand side, and putting everything else on the right-hand side, gives us the formal wave equation

$$\Box h^{\alpha\beta} = -\frac{16\pi G}{c^4}\tau^{\alpha\beta} \tag{6.51}$$

for the potentials $h^{\alpha\beta}$, where

$$\tau^{\alpha\beta} := (-g)\big(T^{\alpha\beta}[\mathsf{m}, g] + t^{\alpha\beta}_{\rm LL}[h] + t^{\alpha\beta}_{\rm H}[h]\big) \tag{6.52}$$

is the *effective energy-momentum pseudotensor* for the wave equation. We have introduced

$$(-g)t^{\alpha\beta}_{\rm H} := \frac{c^4}{16\pi G}\Big(\partial_\mu h^{\alpha\nu}\partial_\nu h^{\beta\mu} - h^{\mu\nu}\partial_{\mu\nu}h^{\alpha\beta}\Big) \tag{6.53}$$

as an additional (harmonic-gauge) contribution to the effective energy-momentum pseudotensor.

In our expression for $\tau^{\alpha\beta}$ we have indicated that the energy-momentum tensor $T^{\alpha\beta}$ is a functional of matter variables m, in addition to being a functional of the metric tensor $g_{\alpha\beta}$ (which is obtained from the gravitational potentials). As an example, when the matter consists of a perfect fluid, m collectively denotes variables such as the mass density ρ, pressure p, and velocity field u^α. We have also indicated that the Landau–Lifshitz and harmonic pseudotensors are functionals of $h^{\alpha\beta}$. As we shall see below, imposition of the

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gauge conditions (6.49) is equivalent to enforcing the conservation equations

$$\partial_\beta \tau^{\alpha\beta} = 0, \tag{6.54}$$

which can be compared with Eq. (6.8). It is easy to verify that $(-g)t_{\rm H}^{\alpha\beta}$ is separately conserved, in that it satisfies $\partial_\beta[(-g)t_{\rm H}^{\alpha\beta}] = 0$ as an identity.

The wave equation of Eq. (6.51) is the main starting point of post-Minkowskian theory. It is worth emphasizing the fact that this equation, together with Eq. (6.49) or (6.54), is an *exact formulation* of the Einstein field equations; no approximations have been introduced at this stage.

For a metric $g_{\alpha\beta}$ to satisfy the complete set of Einstein field equations, it is necessary for the potentials $h^{\alpha\beta}$ to satisfy *both* the wave equation *and* the gauge condition/conservation statement; it is the *union* of Eq. (6.51) and (6.49) or (6.54) that is equivalent to the original form of the Einstein field equations, $G^{\alpha\beta} = (8\pi G/c^4)T^{\alpha\beta}$. The two sets of equations play different roles. The wave equation (6.51) determines the gravitational potentials $h^{\alpha\beta}[\mathsf{m}]$ (and therefore the metric) as functions of the harmonic coordinates x^α, in terms of the matter variables m; these, however, remain undetermined until we also involve the conservation equation (6.54). It is this equation that determines the behavior of the matter variables in a curved spacetime whose metric is built from $h^{\alpha\beta}[\mathsf{m}]$. Solving both sets of equations therefore determines both the metric *and* the matter variables. This reminds us of John Wheeler's famous words: *matter tells spacetime how to curve, and spacetime tells matter how to move*; the decomposition of the field equations into a wave equation and a gauge condition/conservation statement provides a mathematical representation of this maxim.

We have just seen that when the complete set of Einstein field equations is integrated, one cannot solve for the metric independently of the matter variables, and one cannot solve for the matter variables independently of the metric. It is useful to observe, however, that when the equations are decomposed into the subsets [wave equation] and [gauge condition/conservation statement], one is entirely free to solve the wave equation (6.51) without also enforcing the gauge condition of Eq. (6.49) or the conservation statement of Eq. (6.54). Solving the wave equation independently of the gauge condition/conservation statement amounts to integrating only a subset of the Einstein field equations, and the procedure returns ten gravitational potentials $h^{\alpha\beta}[\mathsf{m}]$ expressed as functionals of undetermined matter variables m. The metric obtained from these potentials is also a functional of m, and it is not yet a solution to the Einstein field equations; it becomes a solution only when the gauge condition/conservation statement is imposed as an additional condition on the matter variables. The wave equation (6.51), taken by itself independently of Eqs. (6.49) or (6.54), is known as the *relaxed Einstein field equation*.

Box 6.3 Wave equation in flat and curved spacetimes

Because it involves second derivatives of the potentials, the term $h^{\mu\nu}\partial_{\mu\nu}h^{\alpha\beta}$ on the right-hand side of the field equations might have been more appropriately placed on the left-hand side, and joined together with the wave-operator term. In fact, there is a way of combining all second-order derivatives into a *curved-spacetime* wave operator. For this purpose we treat $h^{\alpha\beta}$ as a collection of ten scalar fields instead of as a tensor field.

The scalar wave operator associated with the metric $g_{\alpha\beta}$ (which is to be constructed from the potentials) is denoted $\Box_g$, and it has the following action on each of the ten potentials:

$$\begin{aligned}\Box_g h^{\alpha\beta} &= \frac{1}{\sqrt{-g}}\partial_\mu\bigl(\sqrt{-g}g^{\mu\nu}\partial_\nu h^{\alpha\beta}\bigr)\\ &= \frac{1}{\sqrt{-g}}\partial_\mu\Bigl[\bigl(\eta^{\mu\nu} - h^{\mu\nu}\bigr)\partial_\nu h^{\alpha\beta}\Bigr]\\ &= \frac{1}{\sqrt{-g}}\Bigl[\Box h^{\alpha\beta} - h^{\mu\nu}\partial_{\mu\nu}h^{\alpha\beta}\Bigr], \end{aligned} \tag{6.55}$$

where we have used the harmonic gauge conditions in the last step. This expression does indeed involve all second-derivative terms that appear in Eq. (6.51). The field equations could then be formulated in terms of $\Box_g$, and this was, in fact, the approach adopted by Kovacs and Thorne in their series of papers on the generation of gravitational waves. This approach, while conceptually compelling, is not as immediately useful for post-Minkowskian theory as the approach adopted here, which is based on the Minkowski wave operator. It is indeed much simpler to solve the wave equation in flat spacetime than it is to solve it in a curved spacetime with a complicated (and as yet unknown) metric.

6.2.2 Formal solution to the wave equation

The wave equation of Eq. (6.51) admits the formal solution

$$h^{\alpha\beta}(x) = \frac{4G}{c^4}\int G(x,x')\tau^{\alpha\beta}(x')\,d^4x', \tag{6.56}$$

where $x = (ct, \boldsymbol{x})$ is a field point and $x' = (ct', \boldsymbol{x}')$ a source point. The two-point function $G(x, x')$ is the *retarded Green's function* of the Minkowski wave operator, which satisfies

$$\Box G(x,x') = -4\pi\delta(x - x'), \tag{6.57}$$

and which is known to be a function of $x - x'$ only. (An explicit expression will be presented in Sec. 6.3.) This property is sufficient to prove that if the effective energy-momentum pseudotensor $\tau^{\alpha\beta}$ satisfies the conservation identities of Eq. (6.54), then the potentials $h^{\alpha\beta}$ will satisfy the harmonic gauge conditions of Eq. (6.49). The converse property, that $\partial_\beta\tau^{\alpha\beta} = 0$ when $\partial_\beta h^{\alpha\beta} = 0$, follows immediately from the wave equation (6.51).

To prove that $\partial_\beta h^{\alpha\beta} = 0$ when $\partial_\beta\tau^{\alpha\beta} = 0$, we begin by differentiating Eq. (6.56) with respect to x^β:

$$\partial_\beta h^{\alpha\beta} = \frac{4G}{c^4}\int \partial_\beta G(x,x')\tau^{\alpha\beta}(x')\,d^4x'. \tag{6.58}$$

Using the previously mentioned property that $G(x, x')$ depends on $x - x'$ only, we may write this as

$$\partial_\beta h^{\alpha\beta} = \frac{4G}{c^4}\int \bigl[-\partial_{\beta'}G(x,x')\bigr]\,\tau^{\alpha\beta}(x')\,d^4x', \tag{6.59}$$

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in which the Green's function is now differentiated with respect to x'^{β}. Integrating by parts, we arrive at

$$\partial_\beta h^{\alpha\beta} = \frac{4G}{c^4} \int G(x, x') \partial_{\beta'} \tau^{\alpha\beta}(x')\, d^4x'. \tag{6.60}$$

This equation reveals directly that $h^{\alpha\beta}$ satisfies the harmonic gauge conditions when $\tau^{\alpha\beta}$ is conserved.

6.2.3 Iteration of the relaxed field equations

The question that concerns us now is this: given the complexity of Eqs. (6.51)–(6.54), how can we construct solutions for a particular choice of matter variables m? Our answer will be: by successive approximations. We shall not attempt to find exact solutions to our equations; instead, we shall retreat to an approximate context in which our spacetime deviates only moderately from Minkowski spacetime. To construct the metric of this spacetime we consider a formal expansion of the form

$$h^{\alpha\beta} = Gk_1^{\alpha\beta} + G^2 k_2^{\alpha\beta} + G^3 k_3^{\alpha\beta} + \cdots \tag{6.61}$$

for the gravitational potentials. Such an expansion in powers of G is known as a *post-Minkowskian expansion*, and our hope is that the expansion – an asymptotic expansion that is not expected to converge – will give rise to an acceptable approximation to the true metric, at least in a useful portion of the spacetime. In the mathematical language of asymptotic expansions, our hope is that $g_{\alpha\beta}(x) - g^n_{\alpha\beta}(x) = O(G^{n+1})$ when x is within a wide domain $\mathscr{U}$ of the spacetime manifold; here $g^n_{\alpha\beta}$ is the metric obtained from Eq. (6.61) after truncating the asymptotic series to order G^n. Equation (6.61) gives rise to the successive approximations $h_0^{\alpha\beta} = 0$, $h_1^{\alpha\beta} = Gk_1^{\alpha\beta}$, $h_2^{\alpha\beta} = Gk_1^{\alpha\beta} + G^2 k_2^{\alpha\beta}$, and so on, for the gravitational potentials.

Box 6.4 The expansion parameter G

This development in powers of G is a formal device only. Because G has dimensions, its numerical value depends on the units in which it is evaluated, and it seems ridiculous to let it play the role of a "small" expansion parameter. For example, we were raised in geometrized units in which $G = 1$, and this does not look like a small quantity. The actual expansion parameter in a typical situation involving a characteristic mass m_c confined to a region of characteristic size r_c is $Gm_c/(c^2 r_c)$, which is small in situations involving weak gravitational fields. Because the proper specification of the expansion parameter requires additional information that is specific to each situation considered, it is economical to stick with G as a formal expansion parameter, and let each physical situation dictate the translation to a meaningful, dimensionless parameter. The absence of a unique, dimensionless expansion parameter for the Einstein field equations is part of the reason why the expansions of post-Minkowskian and post-Newtonian theory are believed to be asymptotic sequences that may not converge.

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In principle we might begin the process of solving the Einstein field equations by substituting Eq. (6.61) into Eq. (6.51) and plucking out terms that share the same power of G. In practice, however, it is more convenient to proceed by iterations, as we now explain.

In the *zeroth iteration* of the relaxed field equations we set $h_0^{\alpha\beta} = 0$ and immediately get $g^0_{\alpha\beta} = \eta_{\alpha\beta}$, the metric of Minkowski spacetime. From this we construct $T^{\alpha\beta}[\mathsf{m}, g] = T^{\alpha\beta}[\mathsf{m}, \eta]$, $t^{\alpha\beta}_{\rm LL}[h] = t^{\alpha\beta}_{\rm LL}[h_0] = 0$, and $t^{\alpha\beta}_{\rm H}[h] = t^{\alpha\beta}_{\rm H}[h_0] = 0$. From all this we obtain $\tau_0^{\alpha\beta} = T^{\alpha\beta}[\mathsf{m}, \eta]$; this is the energy-momentum tensor of the matter variables m, and in the zeroth iteration these live in Minkowski spacetime.

In the *first iteration* of the relaxed field equations we solve the wave equation $\Box h^{\alpha\beta} = -(16\pi G/c^4)\tau_0^{\alpha\beta}$ for $h_1^{\alpha\beta} = Gk_1^{\alpha\beta}$. Because the source $\tau_0^{\alpha\beta}$ is known from the zeroth iteration, the wave equation can be integrated without difficulty (at least in principle), and this returns the potentials $h_1^{\alpha\beta}$ as functionals of the matter variables m, which have yet to be determined. From the potentials we form the metric $g^1_{\alpha\beta}$ and construct $\tau_1^{\alpha\beta}$, an improved version of the effective energy-momentum pseudotensor. This involves the material contribution $T^{\alpha\beta}[\mathsf{m}, g_1]$, as well as the field contributions $t^{\alpha\beta}_{\rm LL}[h_1]$ and $t^{\alpha\beta}_{\rm H}[h_1]$.

In the *second iteration* of the relaxed field equations we solve the wave equation $\Box h^{\alpha\beta} = -(16\pi G/c^4)\tau_1^{\alpha\beta}$ for $h_2^{\alpha\beta} = Gk_1^{\alpha\beta} + G^2 k_2^{\alpha\beta}$, an improved version of the gravitational potentials. Because the source $\tau_1^{\alpha\beta}$ is known from the first iteration, the wave equation can once more be integrated, and $h_2^{\alpha\beta}$ are again functionals of the undetermined matter variables m. From the new potentials we form the metric $g^2_{\alpha\beta}$ and construct $\tau_2^{\alpha\beta}$, the latest version of the effective energy-momentum pseudotensor. The stage is ready for the next iteration.

After n iterations we obtain the potentials $h_n^{\alpha\beta} = Gk_1^{\alpha\beta} + G^2k_2^{\alpha\beta} + \cdots + G^n k_n^{\alpha\beta}$, the nth post-Minkowskian approximation to the true potentials $h^{\alpha\beta}$. These functions of the harmonic coordinates x^α are functionals of the matter variables m, which must now be determined. This is accomplished in the very last step of the procedure, the implementation of the gauge condition/conservation statement, which has not yet been invoked. We thus impose $\partial_\beta h_n^{\alpha\beta} = 0$ on our iterated solution to the relaxed field equations; this determines m and returns $g^n_{\alpha\beta}(x)$ as a proper tensor field in spacetime. Equivalently, we may enforce the conservation equation $\partial_\beta \tau_{n-1}^{\alpha\beta} = 0$, which (as we have seen) is formally equivalent to $\partial_\beta h_n^{\alpha\beta} = 0$. It is important to observe that while the gauge condition involves $h_n^{\alpha\beta}$, the conservation statement involves $\tau_{n-1}^{\alpha\beta}$; these quantities are linked by the iteration procedure described previously.

Let us illustrate the foregoing discussion by choosing the matter content of the spacetime to consist of N point masses labeled by an index $A = (1, 2, \ldots, N)$. In this case the collective matter variables m denote the set of vectors $\boldsymbol{r}_A(t)$, which give the position of each body in the harmonic system of coordinates. After n iterations of the relaxed field equations we obtain gravitational potentials of the form $h_n^{\alpha\beta}(x^\alpha; \boldsymbol{r}_A)$; these are functions of the spacetime coordinates x^α and functionals of the trajectories $\boldsymbol{r}_A(t)$. At this stage of the procedure the trajectories are not determined; the functions $\boldsymbol{r}_A(t)$ are completely arbitrary. In the final step we enforce the conservation equation $\partial_\beta \tau_{n-1}^{\alpha\beta} = 0$, and this produces equations

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of motion of the form

$$\frac{d^2 \boldsymbol{r}_A}{dt^2} = O(G) + O(G^2) + \cdots + O(G^{n-1}) . \tag{6.62}$$

These are used to determine $\boldsymbol{r}_A(t)$, and the task is completed: we have the metric and the motion of the individual bodies. These considerations indicate that two iterations of the relaxed field equations are required to obtain the Newtonian equations of motion – the $O(G)$ term on the right-hand side of Eq. (6.62).

It is important to understand that the iterations must be performed on the relaxed equations only, and not on the full set of Einstein field equations. In other words, one iterates the wave equation only, and leaves the gauge condition/conservation statement alone, until the final iteration is carried out; the gauge condition/conservation statement is enforced in the very last step of the procedure. It would indeed be misguided to enforce it at every iterative step. To see why, imagine that we choose to enforce $\partial_\beta \tau^{\alpha\beta} = 0$ immediately at the zeroth iteration. Because $\tau_0^{\alpha\beta} = T^{\alpha\beta}[\mathsf{m}, \eta]$, this is the conservation equation for matter fields in Minkowski spacetime, and it implies that the matter cannot be subjected to gravitational interactions. (In the illustrative case of point masses examined previously, the bodies would have to move on straight lines.) The next iteration would produce $h_1^{\alpha\beta}$ as sourced by this matter field, and the next version of the conservation statement, $\partial_\beta \tau_1^{\alpha\beta} = 0$, would imply that the matter is, after all, subjected to a gravitational interaction. (In our example, the point masses would now be allowed to move according to the Newtonian equations of motion, in a gravitational field determined as if the masses were moving on straight lines.) We have a contradiction, and this tension is best avoided by delaying the implementation of the gauge condition/conservation statement until the very last step of the iterative procedure.

As a small technical point, we might mention that the procedure does retain a limited amount of latitude. As described above, the penultimate step in the iterative procedure is to solve the wave equation $\Box h^{\alpha\beta} = -(16\pi G/c^4)\tau_{n-1}^{\alpha\beta}$ for $h_n^{\alpha\beta}$, given the known source $\tau_{n-1}^{\alpha\beta}$. The last step is to impose the additional conditions $\partial_\beta \tau_{n-1}^{\alpha\beta} = 0$. These steps can be switched: once $\tau_{n-1}^{\alpha\beta}$ is constructed from $h_{n-1}^{\alpha\beta}$ in the $(n-1)$th iteration, one can immediately enforce the conservation equation $\partial_\beta \tau_{n-1}^{\alpha\beta} = 0$. The final step is then to obtain $h_n^{\alpha\beta}$ by integrating the wave equation, and the gauge condition $\partial_\beta h_n^{\alpha\beta} = 0$ will automatically be satisfied by the solution.

We can be even more flexible. If we are interested *only* in the equations of motion that arise from the $(n-1)$th iteration, and *not* in the spacetime metric that is generated by that motion, then we do not actually have to complete the iterations to obtain $h_n^{\alpha\beta}$. The solutions $h_{n-1}^{\alpha\beta}$ are sufficient to insert into the conservation equations $\partial_\beta \tau_{n-1}^{\alpha\beta} = 0$, from which the motion of the system can be determined consistently to order G^{n-1}.

The iterative, post-Minkowskian method described in this section is technically demanding to carry out, and in the next chapter we shall develop a number of helpful techniques that permit its successful implementation. Before we start, however, we must learn how to solve a wave equation in flat spacetime. This is the topic of the following section.

6.3 Integration of the wave equation

At first sight the wave equation (6.51) appears to be highly non-linear, with the potentials $h^{\alpha\beta}$ present on both sides of the equation. In Sec. 6.2.3 we outlined an iterative procedure that ensures that in the course of each iteration, the wave equation is actually linear in $h^{\alpha\beta}$ and involves a known source term $\tau^{\alpha\beta}$. The task of solving the relaxed field equations therefore appears to be straightforward, and in this section we introduce a number of powerful techniques to integrate the wave equation.

For simplicity we shall eliminate all unnecessary tensorial indices on the wave equation, which we now write as

$$\Box\psi = -4\pi\mu. \tag{6.63}$$

The scalar potential $\psi(x)$ plays the role of $h^{\alpha\beta}$, and the source function $\mu(x)$ plays the role of $(4G/c^4)\tau^{\alpha\beta}$; the remaining factor of 4π is retained for later convenience. Here $x = (ct, \boldsymbol{x})$ labels a spacetime event, and we recall that

$$\Box := \eta^{\alpha\beta}\partial_{\alpha\beta} = -\frac{1}{c^2}\frac{\partial^2}{\partial t^2} + \nabla^2 \tag{6.64}$$

is the wave operator of Minkowski spacetime. The source function $\mu(x)$ is assumed to be known, but unlike the typical situations encountered in electrodynamics, for example, it cannot be assumed to be confined to a bounded region of three-dimensional space; it is instead taken to be distributed over all space. The reason originates from the post-Minkowskian context: as we have seen, during each iteration of the relaxed field equations, $\tau^{\alpha\beta}$ is built in part from $T^{\alpha\beta}$, which normally has compact support, and in part from $t^{\alpha\beta}_{\rm LL}$ and $t^{\alpha\beta}_{\rm H}$, which do not because they are constructed from $h^{\alpha\beta}$, which extends over all space. Our source term in Eq. (6.63) will therefore extend over all space, but μ is assumed to fall off sufficiently rapidly to ensure that ψ decays at least as fast as r^{-1} (where $r := |\boldsymbol{x}|$). Occasionally we shall find it useful to decompose μ into a piece $\mu_{\rm c}$ with compact support (analogous to $T^{\alpha\beta}$) and a piece $\mu_{\rm nc}$ with non-compact support.

A summary of our main results in this section is contained in Box 6.7.

6.3.1 Retarded Green's function

The central tool to integrate Eq. (6.63) is the *retarded Green's function* $G(x, x')$, a solution to

$$\Box G(x, x') = -4\pi\delta(x - x') = -4\pi\delta(ct - ct')\delta(\boldsymbol{x} - \boldsymbol{x}'), \tag{6.65}$$

with the property that $G(x, x')$ vanishes when x is in the past of x'. As we show in Box 6.5, the Green's function is given explicitly by

$$G(x, x') = \frac{\delta(ct - ct' - |\boldsymbol{x} - \boldsymbol{x}'|)}{|\boldsymbol{x} - \boldsymbol{x}'|}, \tag{6.66}$$

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where

$$|\boldsymbol{x} - \boldsymbol{x}'| := \sqrt{(x - x')^2 + (y - y')^2 + (z - z')^2} \tag{6.67}$$

is the Euclidean distance between the field point $\boldsymbol{x}$ and the source point $\boldsymbol{x}'$. Alternatively, the Green's function can be expressed as

$$G(x, x') = 2\Theta(ct - ct')\,\delta\big[(ct - ct')^2 - |\boldsymbol{x} - \boldsymbol{x}'|^2\big], \tag{6.68}$$

in terms of the flat spacetime interval Δs^2 between x and x'; here $\Theta(ct - ct')$ is the Heaviside step function, which is equal to one when $ct > ct'$ and zero when $ct < ct'$.

Box 6.5 **Green's function for the wave equation**

To construct a solution to Eq. (6.65) we write the Green's function as the Fourier transform

$$G(x, x') = \frac{1}{2\pi} \int \tilde{G}(k; \boldsymbol{x}, \boldsymbol{x}') e^{-ik(ct-ct')}\, dk, \tag{1}$$

and we represent the time delta function as

$$\delta(ct - ct') = \frac{1}{2\pi} \int e^{-ik(ct-ct')}\, dk.$$

Substituting these expressions into Green's equation yields

$$\big(\nabla^2 + k^2\big)\tilde{G}(k; \boldsymbol{x}, \boldsymbol{x}') = -4\pi\,\delta(\boldsymbol{x} - \boldsymbol{x}'). \tag{2}$$

When $k = 0$ this equation reduces to Green's equation for the Poisson equation, and from this comparison we learn that $\tilde{G}(0; \boldsymbol{x}, \boldsymbol{x}') = |\boldsymbol{x} - \boldsymbol{x}'|^{-1}$.

We can anticipate that for $k \neq 0$, $\tilde{G}$ will be of the form

$$\tilde{G}(k; \boldsymbol{x}, \boldsymbol{x}') = \frac{g(k, |\boldsymbol{x} - \boldsymbol{x}'|)}{|\boldsymbol{x} - \boldsymbol{x}'|}, \tag{3}$$

with g representing a function that stays non-singular when the second argument, $R := |\boldsymbol{x} - \boldsymbol{x}'|$, approaches zero. That $\tilde{G}$ should depend on the spatial variables through R only can be justified on the grounds that three-dimensional space is both homogeneous (so that $\tilde{G}$ can depend only on the vector $\boldsymbol{R} := \boldsymbol{x} - \boldsymbol{x}'$) and isotropic (so that only the length of the vector matters, and not its direction). That $\tilde{G}$ should behave as $1/R$ when R is small is justified by the following discussion.

We take Eq. (2) and integrate both sides over a sphere of small radius ε centered at $\boldsymbol{x}'$. Since $\nabla^2\tilde{G} = \boldsymbol{\nabla} \cdot \boldsymbol{\nabla}\tilde{G}$, we can use Gauss's theorem to get

$$\oint_{R=\varepsilon} \boldsymbol{\nabla}\tilde{G} \cdot d\boldsymbol{S} + k^2 \int_{R<\varepsilon} \tilde{G}\, d^3x = -4\pi,$$

where $d\boldsymbol{S}$ is the surface element on the sphere. In this equation, the volume integral is of order $\tilde{G}\varepsilon^3$ and it contributes nothing in the limit $\varepsilon \to 0$, unless $\tilde{G}$ happens to be as singular as $1/\varepsilon^3$. The surface integral, on the other hand, is equal to

$$4\pi\varepsilon^2 \frac{d\tilde{G}}{dR}\bigg|_{R=\varepsilon}.$$

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If $\tilde{G}$ were to behave as $1/\varepsilon^3$, then $d\tilde{G}/dR$ would be of order $1/\varepsilon^4$, the surface integral would contribute a term of order $1/\varepsilon^2$, and the left-hand side could never give rise to the required -4π. We conclude that $\tilde{G}$ cannot be so singular, and that the left-hand side is dominated by the surface integral. This implies that $\tilde{G}$ must be of order $1/\varepsilon$, as was anticipated in Eq. (3). Setting $\tilde{G} = g/R$ returns $-4\pi g(k, \varepsilon) + O(\varepsilon)$ for the surface integral, and this gives us the condition $g(k, 0) = 1$. We also recall that $g(0, R) = 1$.

We may now safely take $R \neq 0$ and substitute Eq. (3) into Eq. (2), taking its right-hand side to be zero. Since $\tilde{G}$ depends on $\boldsymbol{x}$ only through R, the Laplacian operator becomes

$$\nabla^2 \to \frac{1}{R^2}\frac{d}{dR}R^2\frac{d}{dR}.$$

Acting with this on $\tilde{G} = g/R$ yields g''/R and Eq. (2) becomes

$$g'' + k^2 g = 0,$$

with a prime indicating differentiation with respect to R. With the boundary condition at $R = 0$ specified previously, two linearly independent solutions to this equation are

$$g_\pm(k, R) = e^{\pm ikR}.$$

Substituting this into Eq. (3), and that into Eq. (1), we obtain

$$G_\pm(x, x') = \frac{1}{2\pi}\int \frac{e^{\pm ikR}}{R} e^{-ik(ct-ct')}\, dk = \frac{1}{2\pi R}\int e^{-ik(ct-ct'\mp R)}\, dk,$$

or

$$G_\pm(x, x') = \frac{\delta\big(ct - ct' \mp |\boldsymbol{x} - \boldsymbol{x}'|\big)}{|\boldsymbol{x} - \boldsymbol{x}'|}. \tag{4}$$

The function $G_+(x, x')$, which is non-zero when $ct - ct' = +R$, is known as the *retarded Green's function*; the function $G_-(x, x')$, which is non-zero when $ct - ct' = -R$, is known as the *advanced Green's function*.

The retarded Green's function can be expressed in the alternative form

$$G_+(x, x') = 2\Theta(ct - ct')\delta\big[(ct - ct')^2 - |\boldsymbol{x} - \boldsymbol{x}'|^2\big]. \tag{5}$$

The new argument of the delta function factorizes as $(ct - ct' - R)(ct - ct' + R)$, and when $c(t - t') > 0$ only the first factor may go through zero; the second factor is then equal to $2R$, and the delta function is distributionally equal to $\delta(ct - ct' - R)/(2R)$. At this stage the step function becomes redundant, because the delta function is active only when $c(t - t') > 0$, and we have reproduced Eq. (4).

Similarly, the advanced Green's function can be expressed as

$$G_-(x, x') = 2\Theta(ct' - ct)\delta\big[(ct - ct')^2 - |\boldsymbol{x} - \boldsymbol{x}'|^2\big].$$

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In terms of the retarded Green's function $G(x, x')$, the solution to Eq. (6.63) is

$$\psi(x) = \int G(x, x')\mu(x')\, d^4x', \tag{6.69}$$

where $d^4x' = d(ct')d^3x'$. After substitution of Eq. (6.66) and integration over $d(ct')$, this becomes

$$\psi(t, \boldsymbol{x}) = \int \frac{\mu(t - |\boldsymbol{x} - \boldsymbol{x}'|/c, \boldsymbol{x}')}{|\boldsymbol{x} - \boldsymbol{x}'|} d^3x'. \tag{6.70}$$

This is the *retarded solution* to the wave equation, and the domain of integration extends over $\mathscr{C}(x)$, the *past light cone* of the field point $x = (ct, \boldsymbol{x})$.

6.3.2 Near zone and wave zone: slow-motion condition

In the following subsection the domain $\mathscr{C}(x)$ will be partitioned into a *near-zone domain* $\mathscr{N}$ and a *wave-zone domain* $\mathscr{W}$. Our task in this subsection is to introduce the important notions of near and wave zones in the general context of the wave equation (6.63).

To do so we introduce the following scaling quantities:

$$t_c := \text{characteristic time scale of the source}, \tag{6.71a}$$

$$\omega_c := \frac{2\pi}{t_c} = \text{characteristic frequency of the source}, \tag{6.71b}$$

$$\lambda_c := \frac{2\pi c}{\omega_c} = ct_c = \text{characteristic wavelength of the radiation}. \tag{6.71c}$$

The characteristic time scale t_c is the time required for noticeable changes to occur within the source; it is defined such that $\partial_t \mu$ is typically of order μ/t_c over the support of the source function. The characteristic frequency ω_c and wavelength λ_c are derived directly from t_c. If, for example, μ oscillates with a frequency ω, then $t_c \sim 2\pi/\omega$, $\omega_c \sim \omega$, and $\lambda_c \sim 2\pi c/\omega$.

The near zone and the wave zone are defined as

$$\text{near zone:} \quad r \ll \lambda_c = \frac{2\pi c}{\omega_c} = ct_c, \tag{6.72a}$$

$$\text{wave zone:} \quad r \gg \lambda_c = \frac{2\pi c}{\omega_c} = ct_c. \tag{6.72b}$$

Thus, the near zone is the region of three-dimensional space in which $r := |\boldsymbol{x}|$ is small compared with a characteristic wavelength λ_c, while the wave zone is the region in which r is large compared with this length scale. As we can see from the example of Box 6.6, the potential behaves very differently in the two zones: in the near zone the difference between $\tau := t - r/c$ and t is small (the field retardation is unimportant), and time derivatives are small compared with spatial derivatives; in the wave zone the difference between $\tau = t - r/c$ and t is large, and time derivatives are comparable to spatial derivatives. These properties are shared by all generic solutions to the wave equation.

Another important feature of the near zone concerns the quantity $(r/c)\partial_t\mu$. This is of order $(r/c)(\mu/t_c)$, or $(r/\lambda_c)\mu$, which is much smaller than μ. In the near zone, therefore,

$$\frac{r}{c}\frac{\partial\mu}{\partial t} = O\Big(\frac{r}{\lambda_c}\mu\Big) \ll \mu. \tag{6.73}$$

This states, simply, that the source retardation is unimportant within the near zone.

Thus far our considerations have been general, and our definitions of near and wave zones apply whether the source function μ is extended over all space or confined to a bounded region V. In addition, our definitions apply independently of the existence of a slow-motion condition, to which we turn next.

When the source function μ has a piece μ_c with compact support, we can introduce the additional scaling quantities

$$r_c := \text{characteristic length scale of the compact-support source,} \tag{6.74a}$$

$$v_c := \frac{r_c}{t_c} = \text{characteristic velocity within the source.} \tag{6.74b}$$

The characteristic radius r_c is defined such that μ_c vanishes outside a sphere of radius r_c; this part of μ has support only within this sphere. The characteristic velocity v_c is defined in terms of the scales r_c and t_c; it represents the speed with which changes in the source propagate across the region of space occupied by the source. In the case of a fluid, for example, v_c would be associated with the speed of sound within the fluid. In a binary-star system, v_c would be associated with the orbital velocities of the stars.

A *slow-motion condition* is in effect when the characteristic velocity v_c is small compared with the speed of light:

$$v_c \ll c \qquad \text{(slow-motion condition).} \tag{6.75}$$

It then follows from Eq. (6.75) that

$$r_c \ll \lambda_c \qquad \text{(slow-motion condition);} \tag{6.76}$$

this equation states that μ_c is necessarily situated deep within the near zone when a slow-motion condition is in effect.

Box 6.6 Dipole solution to the wave equation

We examine the solution to a specific version of Eq. (6.63),

$$\psi = (\boldsymbol{p}\cdot\boldsymbol{n})\bigg[\frac{\cos\omega(t-r/c)}{r^2} - \frac{\omega}{c}\frac{\sin\omega(t-r/c)}{r}\bigg],$$

which corresponds to $\mu = -\boldsymbol{p}\cdot\boldsymbol{\nabla}\delta(\boldsymbol{x})\cos\omega t$. Here $\boldsymbol{p}$ is a constant vector, $r := |\boldsymbol{x}|$, $\boldsymbol{n} := \boldsymbol{x}/r$ is the unit radial vector, and ω is an angular frequency. Physically speaking, this solution represents the scalar potential of a dipole of constant direction $\boldsymbol{p}$, oscillating in strength with a frequency $f = \omega/(2\pi)$; the wavelength of the radiation produced by the oscillating dipole is $\lambda = c/f = 2\pi c/\omega$.

Our first observation is that ψ behaves very differently depending on whether r is small or large compared with λ. When $r \ll \lambda = 2\pi c/\omega$, the trigonometric functions can be expanded in powers of $\omega r/c$, and the result is

$$\psi = (\boldsymbol{p}\cdot\boldsymbol{n})\frac{\cos\omega t}{r^2}\left[1 + O\left(\frac{\omega^2 r^2}{c^2}\right)\right] \qquad \text{(near zone)},$$

with a correction term that is quadratic in $r/\lambda \ll 1$. We observe also that in the *near zone* – the region $r \ll \lambda$ – the derivatives of ψ are related by

$$\frac{\partial_t \psi}{c|\boldsymbol{\nabla}\psi|} = O\left(\frac{\omega r}{c}\right) \qquad \text{(near zone)}.$$

In the near zone, therefore, a time derivative is smaller than a spatial derivative (multiplied by c) by a factor of order $r/\lambda \ll 1$.

When, on the other hand, $r \gg \lambda = 2\pi c/\omega$, it is no longer appropriate to expand the trigonometric functions, and the potential must be expressed as

$$\psi = -(\boldsymbol{p}\cdot\boldsymbol{n})\frac{\omega}{c}\frac{\sin\omega\tau}{r}\left[1 + O\left(\frac{c}{\omega r}\right)\right] \qquad \text{(wave zone)},$$

in terms of the *retarded-time* variable $\tau := t - r/c$; here the difference between τ and t is large, and the correction term is linear in $\lambda/r \ll 1$. We observe also that in the *wave zone* – the region $r \gg \lambda$ – the derivatives of ψ are related by

$$\frac{\partial_t \psi}{c|\boldsymbol{\nabla}\psi|} = O(1) \qquad \text{(wave zone)}.$$

To obtain this result we have used the fact that the spatial dependence contained in $\boldsymbol{n}$ and r^{-1} produces a spatial derivative of fractional order λ/r, while the spatial dependence contained in $\tau = t - r/c$ produces a spatial derivative of order unity. In the wave zone, therefore, a time derivative has the same order of magnitude as a spatial derivative (multiplied by c).

6.3.3 Integration domains

The integral of Eq. (6.70) extends over the past light cone $\mathscr{C}(x)$ of the field point x. To evaluate the integral we partition $\mathscr{C}(x)$ into two pieces, the *near-zone domain* $\mathscr{N}(x)$ and the *wave-zone domain* $\mathscr{W}(x)$. We place the boundary of the near and wave zones at an arbitrarily selected radius $\mathcal{R}$, with $\mathcal{R}$ imagined to be of the same order of magnitude as λ_c, the characteristic wavelength of the radiation emitted by μ. The near zone is then imagined as a three-dimensional ball of radius $\mathcal{R}$ that traces out a world tube $\mathscr{D}$ in spacetime. We let $\mathscr{N}(x)$ be the part of $\mathscr{C}(x)$ where $r' := |\boldsymbol{x}'| < \mathcal{R}$, and we let $\mathscr{W}(x)$ be the part of $\mathscr{C}(x)$ where $r' > \mathcal{R}$. The near-zone and wave-zone domains join together to form the complete light cone of the field point x: $\mathscr{N}(x) + \mathscr{W}(x) = \mathscr{C}(x)$. The domains are illustrated in Fig. 6.1.

We write Eq. (6.70) as

$$\psi(x) = \psi_{\mathscr{N}}(x) + \psi_{\mathscr{W}}(x), \tag{6.77}$$

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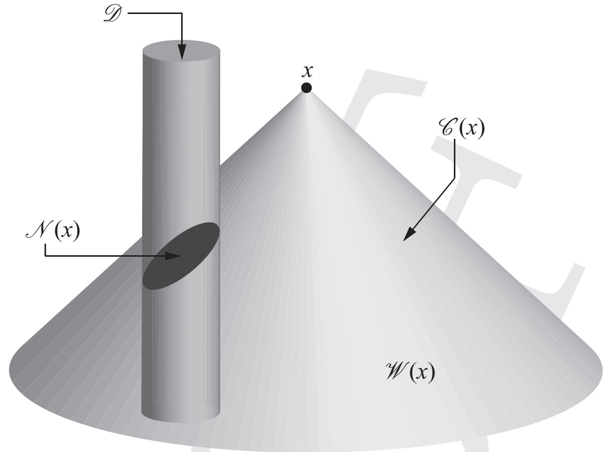

Fig. 6.1 Integration domains for the retarded solution of the wave equation: $\mathscr{C}(x)$ is the past light cone of the field point x; $\mathscr{D}$ is the world tube traced by a three-dimensional ball of radius $\mathcal{R}$, which contains the near-zone region of spacetime; $\mathscr{N}(x)$ is the intersection of $\mathscr{C}(x)$ with the near zone; and $\mathscr{W}(x)$ is the remaining piece of the light cone.

where

$$\psi_{\mathscr{N}}(x) = \int_{\mathscr{N}} G(x, x')\mu(x')\, d^4x' \tag{6.78}$$

is the near-zone portion of the light-cone integral, while

$$\psi_{\mathscr{W}}(x) = \int_{\mathscr{W}} G(x, x')\mu(x')\, d^4x' \tag{6.79}$$

is the wave-zone portion. Methods to evaluate $\psi_{\mathscr{N}}$ and $\psi_{\mathscr{W}}$ will be developed in the following two subsections. It is an important fact that while $\psi_{\mathscr{N}}$ and $\psi_{\mathscr{W}}$ will individually depend on the cutoff parameter $\mathcal{R}$, their sum $\psi = \psi_{\mathscr{N}} + \psi_{\mathscr{W}}$ will necessarily be independent of $\mathcal{R}$. The $\mathcal{R}$-dependence of $\psi_{\mathscr{N}}$ and $\psi_{\mathscr{W}}$ is therefore unimportant, and it can freely be ignored. This observation will serve as a helpful simplifying tool in many subsequent computations.

6.3.4 Integration over the near zone

In this subsection we develop methods to evaluate

$$\psi_{\mathscr{N}}(x) = \int_{\mathscr{N}} \frac{\mu(t - |\boldsymbol{x} - \boldsymbol{x}'|/c, \boldsymbol{x}')}{|\boldsymbol{x} - \boldsymbol{x}'|} d^3x', \tag{6.80}$$

the near-zone contribution to the complete solution $\psi = \psi_{\mathscr{N}} + \psi_{\mathscr{W}}$ to the wave equation. We recall that the domain of integration $\mathscr{N}$ is the intersection between $\mathscr{C}(x)$, the past light cone of the field point x, and the near zone $r' < \mathcal{R}$.

Wave-zone field point

We first evaluate Eq. (6.80) when x is situated in the wave zone, that is, when $r > \mathcal{R}$. For this purpose we introduce a modified integrand,

$$\frac{\mu(t - |\boldsymbol{x} - \boldsymbol{x}'|/c, \boldsymbol{x}')}{|\boldsymbol{x} - \boldsymbol{x}'|} = \int \frac{\mu(t - |\boldsymbol{x} - \boldsymbol{x}'|/c, \boldsymbol{y})}{|\boldsymbol{x} - \boldsymbol{x}'|} \delta(\boldsymbol{y} - \boldsymbol{x}')\, d^3y$$
$$=: \int g(\boldsymbol{x}, \boldsymbol{x}', \boldsymbol{y})\delta(\boldsymbol{y} - \boldsymbol{x}')\, d^3y\,, \tag{6.81}$$

in which we can treat $\boldsymbol{x}'$ and $\boldsymbol{y}$ as independent variables. Knowing that $\boldsymbol{x}'$ lies within the near zone, we treat it as a small vector, and express g as a Taylor expansion about $\boldsymbol{x}' = \boldsymbol{0}$. Keeping just a few terms in this expansion, we have

$$g(\boldsymbol{x}, \boldsymbol{x}', \boldsymbol{y}) = g(\boldsymbol{x}, \boldsymbol{0}, \boldsymbol{y}) + \frac{\partial g}{\partial x'^j} x'^j + \frac{1}{2}\frac{\partial^2 g}{\partial x'^j x'^k} x'^j x'^k + \cdots\,, \tag{6.82}$$

in which all derivatives are evaluated at $\boldsymbol{x}' = \boldsymbol{0}$. But $\partial g/\partial x'^j = -\partial g/\partial x^j$ because g depends on $\boldsymbol{x}'$ only through the combination $|\boldsymbol{x} - \boldsymbol{x}'|$, and our Taylor expansion can be expressed as

$$g(\boldsymbol{x}, \boldsymbol{x}', \boldsymbol{y}) = g(\boldsymbol{x}, \boldsymbol{0}, \boldsymbol{y}) - \frac{\partial g}{\partial x^j} x'^j + \frac{1}{2}\frac{\partial^2 g}{\partial x^j x^k} x'^j x'^k + \cdots \tag{6.83}$$

The derivatives of g are still evaluated at $\boldsymbol{x}' = \boldsymbol{0}$, but because the differentiation is now carried out with respect to $\boldsymbol{x}$, we can set $\boldsymbol{x}' = \boldsymbol{0}$ in g *before* taking the derivatives. Observing that g then becomes a function of $|\boldsymbol{x}| = r$ only, we have

$$g(\boldsymbol{x}, \boldsymbol{x}', \boldsymbol{y}) = g(r, 0, \boldsymbol{y}) - \frac{\partial g(r, 0, \boldsymbol{y})}{\partial x^j} x'^j + \frac{1}{2}\frac{\partial^2 g(r, 0, \boldsymbol{y})}{\partial x^j x^k} x'^j x'^k + \cdots \tag{6.84}$$

Keeping all terms of the Taylor expansion, this is

$$g(\boldsymbol{x}, \boldsymbol{x}', \boldsymbol{y}) = \sum_{\ell=0}^{\infty} \frac{(-1)^\ell}{\ell!} x'^L \partial_L g(r, 0, \boldsymbol{y}), \tag{6.85}$$

where $L := j_1 j_2 \cdots j_\ell$ is a multi-index of the sort introduced back in Sec. 1.5.3. More explicitly, we have established the identity

$$\frac{\mu(t - |\boldsymbol{x} - \boldsymbol{x}'|/c, \boldsymbol{y})}{|\boldsymbol{x} - \boldsymbol{x}'|} = \sum_{\ell=0}^{\infty} \frac{(-1)^\ell}{\ell!} x'^L \partial_L \frac{\mu(t - r/c, \boldsymbol{y})}{r}. \tag{6.86}$$

The dependence of μ/r on the variables x^j is contained entirely within r.

Inserting this within Eq. (6.81) to restore $\boldsymbol{y} = \boldsymbol{x}'$, and substituting the result into Eq. (6.80), we arrive at

$$\psi_{\mathcal{N}}(t, \boldsymbol{x}) = \sum_{\ell=0}^{\infty} \frac{(-1)^\ell}{\ell!} \partial_L \left[\frac{1}{r} \int_{\mathscr{M}} \mu(\tau, \boldsymbol{x}') x'^L\, d^3x' \right], \tag{6.87}$$

where

$$\tau := t - r/c \tag{6.88}$$

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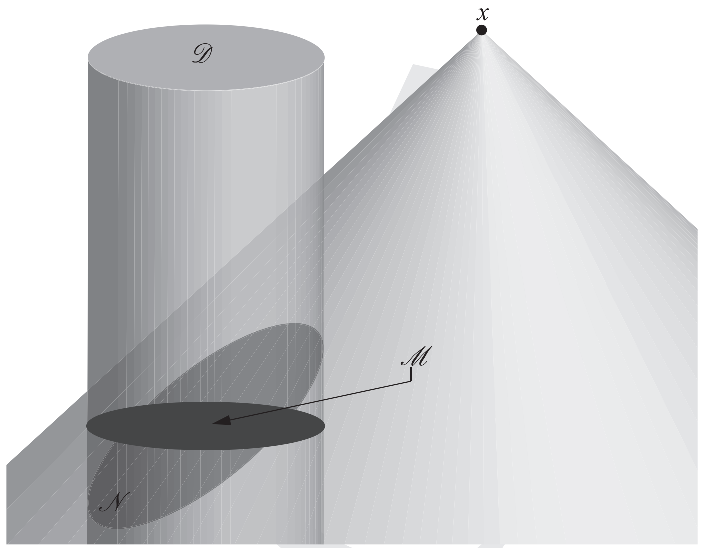

Fig. 6.2 Near-zone integration, wave-zone field point. The domain $\mathscr{M}$ is a surface of constant time bounded externally by the sphere $r' = \mathcal{R}$.

is a retarded-time variable. Note that the temporal dependence of the source function no longer involves $\boldsymbol{x}'$, the variable of integration. The domain of integration has therefore become a surface of constant time (the constant being equal to τ) bounded externally by the sphere $r' = \mathcal{R}$. This domain is denoted $\mathscr{M}$ in Eq. (6.87), and is illustrated in Fig. 6.2.

Equation (6.87) is valid everywhere within the wave zone. It simplifies when $r \to \infty$, that is, when $\psi_{\mathscr{N}}$ is evaluated in the *far-away wave zone*. In this limit we retain only the dominant, r^{-1} term in $\psi_{\mathscr{N}}$, and we approximate Eq. (6.87) by

$$\psi_{\mathscr{N}} = \frac{1}{r} \sum_{\ell=0}^{\infty} \frac{(-1)^\ell}{\ell!} \int_{\mathscr{M}} \partial_L \mu(\tau, \boldsymbol{x}') x'^L \, d^3x' + O(r^{-2}). \tag{6.89}$$

The dependence of μ on x^j is contained in τ, so that $\partial_j \mu = -c^{-1}\mu^{(1)}\partial_j r = -c^{-1}\mu^{(1)} n_j$, in which $\mu^{(1)}$ denotes the first derivative of μ with respect to τ. We used the fact that

$$\partial_j r = n_j, \tag{6.90}$$

where $n^j = x^j/r$ is the unit radial vector. Invoking this result once more, we find that $\partial_{jk}\mu = c^{-2}\mu^{(2)} n_j n_k + O(r^{-1})$, and continuing along these lines reveals that in general, $\partial_L \mu = (-1)^\ell c^{-\ell} \mu^{(\ell)} n_L + O(r^{-1})$. Inserting this into our previous expression for $\psi_{\mathscr{N}}$, we find that Eq. (6.87) becomes

$$\psi_{\mathscr{N}}(t, \boldsymbol{x}) = \frac{1}{r} \sum_{\ell=0}^{\infty} \frac{1}{\ell! c^\ell} n_L \left(\frac{d}{d\tau}\right)^\ell \int_{\mathscr{M}} \mu(\tau, \boldsymbol{x}') x'^L \, d^3x' + O(r^{-2}) \tag{6.91}$$

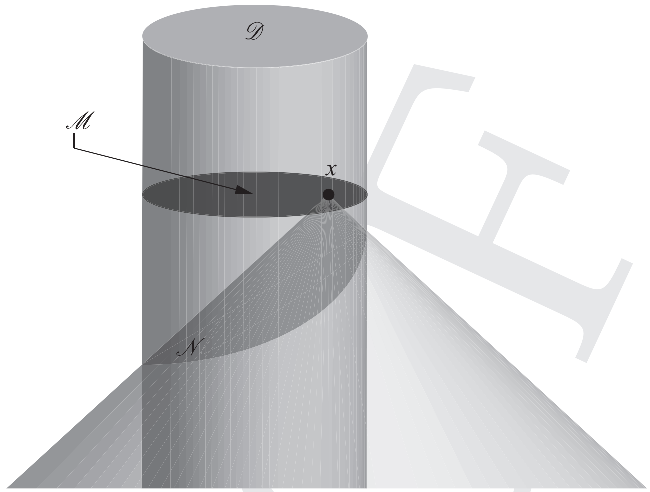

Fig. 6.3 Near-zone integration, near-zone field point.

in the far-away wave zone. This is a *multipole expansion* for the potential $\psi_{\mathscr{N}}$, in which each ℓ-pole moment $\int_{\mathscr{M}} \mu x^L\, d^3x$ is differentiated ℓ-times with respect to τ. Note that $n_L x'^L = n_{j_1} n_{j_2} \cdots n_{j_\ell} x'^{j_1} x'^{j_2} \cdots x'^{j_\ell} = (\boldsymbol{n} \cdot \boldsymbol{x}')^\ell$.

Near-zone field point

We next evaluate Eq. (6.80) when x is situated in the near zone, that is, when $r = |\boldsymbol{x}| < \mathcal{R}$. In this situation, both $\boldsymbol{x}$ and $\boldsymbol{x}'$ lie within the near zone, and $|\boldsymbol{x} - \boldsymbol{x}'|$ can be treated as a small quantity. To evaluate the integral we simply Taylor-expand the time-dependence of the source function,

$$\mu(t - |\boldsymbol{x} - \boldsymbol{x}'|/c) = \mu(t) - \frac{1}{c}\frac{\partial \mu}{\partial t}|\boldsymbol{x} - \boldsymbol{x}'| + \frac{1}{2c^2}\frac{\partial^2 \mu}{\partial t^2}|\boldsymbol{x} - \boldsymbol{x}'|^2 + \cdots,$$

in which all derivatives are evaluated at time t. Substituting this expansion into Eq. (6.80) produces

$$\psi_{\mathscr{N}}(t, \boldsymbol{x}) = \sum_{\ell=0}^{\infty} \frac{(-1)^\ell}{\ell! c^\ell} \left(\frac{\partial}{\partial t}\right)^\ell \int_{\mathscr{M}} \mu(t, \boldsymbol{x}')|\boldsymbol{x} - \boldsymbol{x}'|^{\ell-1}\, d^3x', \tag{6.92}$$

which is valid everywhere within the near zone. Note that once more the domain of integration is $\mathscr{M}$, a surface of constant time bounded externally by the sphere $r' = \mathcal{R}$; here, however, the integral is evaluated at time t instead of at the retarded time τ. The geometry is illustrated in Fig. 6.3.

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6.3.5 Integration over the wave zone

In this subsection we develop a method to evaluate

$$\psi_{\mathscr{W}}(x) = \int_{\mathscr{W}} \frac{\mu(t - |\boldsymbol{x} - \boldsymbol{x}'|/c, \boldsymbol{x}')}{|\boldsymbol{x} - \boldsymbol{x}'|} d^3x', \tag{6.93}$$

the wave-zone portion of the complete solution $\psi = \psi_{\mathscr{N}} + \psi_{\mathscr{W}}$ to the wave equation. We recall that the domain of integration $\mathscr{W}$ is the intersection between $\mathscr{C}(x)$, the past light cone of the field point x, and the wave zone $r' > \mathcal{R}$.

Before we proceed with the work, we pause and ask whether $\psi_{\mathscr{W}}(x)$ could be dispensed with by taking the limit $\mathcal{R} \to \infty$, thereby achieving $\psi_{\mathscr{N}} \to \psi$ and $\psi_{\mathscr{W}} \to 0$. The answer is no: we cannot take $\mathcal{R}$ beyond its original value of order λ_c, and we cannot dispense with $\psi_{\mathscr{W}}$. The reason can be gleaned from Figs. 6.2 and 6.3: The difference between the domain $\mathscr{M}$ and the light cone $\mathscr{C}(x)$ becomes increasingly large as $\mathcal{R}$ increases, and the Taylor expansion for $\mu(t - |\boldsymbol{x} - \boldsymbol{x}'|/c)$ becomes increasingly inaccurate; the resulting expression for $\psi_{\mathscr{N}}$ would then become increasingly unreliable as $\mathcal{R}$ increases beyond λ_c. This lesson was hard learned. Early attempts to integrate the wave equation of post-Minkowskian theory were indeed based on the limit $\mathcal{R} \to \infty$, with the expectation that $\psi_{\mathscr{N}}$ would make a good approximation to ψ. Such attempts led to a host of divergent integrals that had to be argued away or swept under the rug. While these methods could sometimes be teased to give correct physical results, their mathematical justification left a lot to be desired. The decomposition of ψ into near-zone and wave-zone pieces nicely overcomes all these difficulties.

Our method to integrate over $\mathscr{W}$ must reflect the nature of the integrand there, and the fact that we are integrating over a null cone instead of a surface of constant time. For the slow-motion systems that we will generally encounter, the compact-support piece of μ lies deep within the near zone, and therefore vanishes on $\mathscr{W}$. The extended piece survives, and it is built from potentials that are themselves solutions to the wave equation. This implies that for a given integration point $(ct', \boldsymbol{x}')$ on $\mathscr{W}$, $\mu_{\rm nc}$ is predominantly a function of $t' - r'/c$. Integration over the light cone is therefore facilitated by adopting retarded time as a variable of integration. The strategy is therefore this: express the integral of Eq. (6.93) in terms of the spherical coordinates (r', θ', ϕ'), and then switch variables from r' to $u' := ct' - r'$ in order to perform the integration.

The strategy lends itself to a nice geometrical representation (see Fig. 6.4). A surface $u' =$ constant is a future-directed null cone $\mathscr{F}$ that emanates from $r' = 0$. It intersects $\mathscr{C}(x)$ on a two-dimensional surface $\mathscr{S}(u')$ parameterized by the angular variables θ' and ϕ'. Integration on $\mathscr{C}(x)$ can therefore be achieved by integrating over $\mathscr{S}(u')$ and adding the contributions from each relevant $\mathscr{F}$. Integrating on $\mathscr{S}(u')$ amounts to varying θ' and ϕ' over their allowed range, and the integration over $\mathscr{C}(x)$ is completed by varying u', which ranges from $u' = -\infty$ to $u' = u := ct - r$; the final value of u' corresponds to a future null cone that is tangent to $\mathscr{C}(x)$, emanating from the spacetime event at which $r' = 0$ crosses $\mathscr{C}(x)$.

To make these ideas explicit, we first provide a mathematical expression for $\mathscr{S}(u')$. Because $ct' = ct - |\boldsymbol{x} - \boldsymbol{x}'|$ on $\mathscr{C}(x)$ and $ct' = u' + r'$ on $\mathscr{F}$, we find that it is described

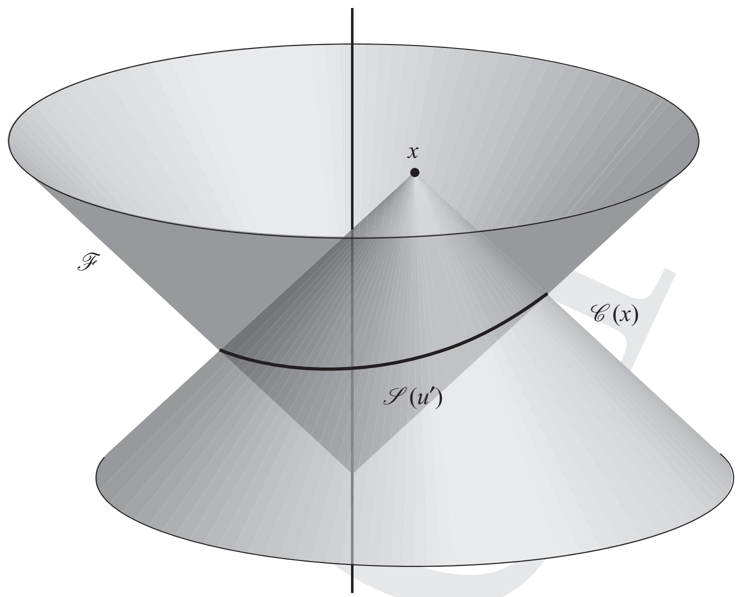

Fig. 6.4 Geometrical representation of the wave-zone integrations. $\mathscr{C}(x)$ is the past light cone of the field point x. $\mathscr{F}$ is the future light cone $u' = ct' - r' =$ constant with apex at $r' = 0$. $\mathscr{S}(u')$ is the two-dimensional surface of intersection between the past and future light cones.

by

$$u' = ct - r' - |\boldsymbol{x} - \boldsymbol{x'}|\,, \tag{6.94}$$

in which u' and t are constant. The equation can be solved for r' expressed as a function of θ' and ϕ':

$$r'(u', \theta', \phi') = \frac{(ct - u')^2 - r^2}{2(ct - u' - \boldsymbol{n'}\cdot\boldsymbol{x})}\,, \tag{6.95}$$

where $\boldsymbol{n'} := \boldsymbol{x'}/r'$. We next return to Eq. (6.93) and change variables from r' to u', using

$$\frac{\partial u'}{\partial r'} = \boldsymbol{n'}\cdot\boldsymbol{\nabla'}u' = \frac{u' - ct + \boldsymbol{n'}\cdot\boldsymbol{x}}{|\boldsymbol{x} - \boldsymbol{x'}|}\,. \tag{6.96}$$

This yields

$$\psi_{\mathscr{W}} = \int_{-\infty}^{u} du' \oint_{\mathscr{S}(u')} \frac{\mu((u' + r')/c, \boldsymbol{x'})}{ct - u' - \boldsymbol{n'}\cdot\boldsymbol{x}} r'(u', \theta', \phi')^2\, d\Omega'\,, \tag{6.97}$$

our new starting expression to calculate the wave-zone contribution to the potential $\psi(x)$.

To proceed it will be necessary to restrict our attention to source functions of the form

$$\mu(x') = \frac{1}{4\pi}\frac{f(\tau')}{r'^n} n'^{\langle L\rangle}, \tag{6.98}$$

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where f is an arbitrary function of $\tau' = t' - r'/c$, n is an arbitrary integer, and $n'^{\langle L\rangle}$ is an STF product of ℓ radial vectors $n'^j = x'^j/r'$; these angular tensors were introduced back in Sec. 1.5.3, and we recall that they are closely related to the spherical-harmonic functions $Y_{lm}(\theta', \phi')$. Fortunately, the restriction imposed here is not too severe from a practical point of view: All source functions to be inserted in wave-zone integrals in this book will be superpositions of the irreducible forms displayed in Eq. (6.98).

Substituting Eq. (6.98) into Eq. (6.97), we obtain

$$\psi_{\mathscr{W}} = \frac{1}{4\pi}\int_{-\infty}^{u} du'\, f(u'/c) \oint_{\mathscr{S}(u')} \frac{n'^{\langle L\rangle}}{r'(u', \theta', \phi')^{n-2}} \frac{d\Omega'}{ct - u' - \boldsymbol{n}'\cdot\boldsymbol{x}}\,. \tag{6.99}$$

The angular integration can be simplified by orienting the coordinate axes so that the selected field point $\boldsymbol{x}$ is aligned with the z-direction, so that $\boldsymbol{n} = \boldsymbol{e}_z$; this specific choice will be undone at the end of our computation. We make use of Eq. (1.164),

$$n'^{\langle L\rangle} = N_\ell \sum_{m=-\ell}^{\ell} \mathscr{Y}_{\ell m}^{\langle L\rangle} Y_{\ell m}(\theta', \phi'), \tag{6.100}$$

where $N_\ell := 4\pi\ell!/(2\ell+1)!!$, integrate over $d\phi'$, and observe that since the rest of the integrand is independent of ϕ', the only surviving term in the sum is $m = 0$. Inserting now $Y_{\ell 0} = [(2\ell+1)/4\pi]^{1/2} P_\ell(\cos\theta')$ and $\mathscr{Y}_{\ell 0}^{\langle L\rangle} = [4\pi/(2\ell+1)]^{1/2} N_\ell^{-1} e_z^{\langle L\rangle}$ within the integral, we obtain

$$\psi_{\mathscr{W}} = \frac{1}{2} n^{\langle L\rangle} \int_{-\infty}^{u} du'\, f(u'/c) \int_{\mathscr{S}(u')} \frac{P_\ell(\xi)}{r'(u', \xi)^{n-2}(ct - u' - r\xi)} d\xi\,, \tag{6.101}$$

in which $\xi := \cos\theta'$ and

$$r'(u', \xi) := r'(u', \theta', 0) = \frac{(ct - u')^2 - r^2}{2(ct - u' - r\xi)}\,. \tag{6.102}$$

Switching integration variables from ξ back to r', using the fact that $\partial\xi/\partial r' = (ct - u' - r\xi)/rr'$, we recast $\psi_{\mathscr{W}}$ in the elegant form

$$\psi_{\mathscr{W}} = \frac{n^{\langle L\rangle}}{2r} \int_{-\infty}^{u} du'\, f(u'/c) \int_{\mathscr{S}(u')} \frac{P_\ell(\xi)}{r'^{(n-1)}} dr'\,, \tag{6.103}$$

in which ξ is now the function of r' determined by Eq. (6.102); an explicit expression will be provided below. We observe that the angular dependence of $\psi_{\mathscr{W}}$ is contained in the factor $n^{\langle L\rangle}$, with $\boldsymbol{n}$ previously chosen to be aligned with the z-direction. But since the remaining integral is now independent of all angles, the orientation of the coordinate axes has become irrelevant, and the special choice $\boldsymbol{n} = \boldsymbol{e}_z$ immaterial; we may now take $\boldsymbol{n}$ to the point in the arbitrary direction specified by the polar angles θ and ϕ. The potential $\psi_{\mathscr{W}}$ has thus become a function of (t, r, θ, ϕ), with the dependence on t contained within $u = ct - r$.

To complete the wave-zone integration we must now give an explicit description of the closed surface $\mathscr{S}(u')$, and specify the limits of the integral over dr' so as to exclude the near zone from the domain of integration. The specific limits depend on whether the field point is in the near zone or in the wave zone.

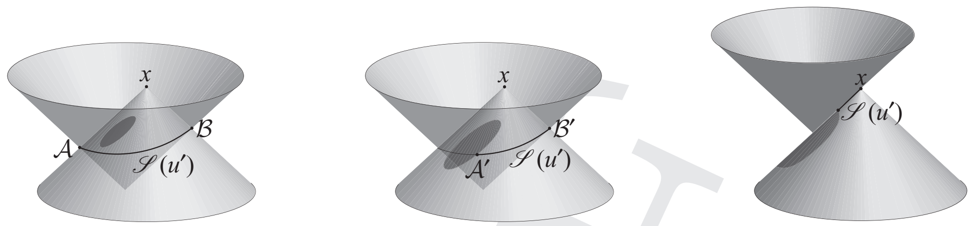

Fig. 6.5 Integration over the domain $\mathscr{W}(x)$, for a field point x in the wave zone, is carried out over each intersection surface $\mathscr{S}(u')$ in a sequence of future null cones $u' =$ constant. The left panel corresponds to $u' < u - 2\mathcal{R}$; the integration runs from $\xi = -1$ (point $\mathcal{A}$) to $\xi = 1$ (point $\mathcal{B}$). The center panel corresponds to $u' > u - 2\mathcal{R}$; the intersection $\mathscr{S}(u')$ terminates at $\mathcal{A}'$, the boundary of the near zone $\mathscr{N}$. The right panel corresponds to $u' = u$; the cones are tangent, and $\mathscr{S}(u')$ runs from the edge of the near zone to x.

Wave-zone field point

To begin we assume that the field point x is situated in the wave zone, so that $r > \mathcal{R}$. We recall that $\mathscr{S}(u')$ is the intersection between the past null cone $\mathscr{C}(x)$ and the future null cone $u' =$ constant. From Fig. 6.5 we see that when $u' < u - 2\mathcal{R}$, $\mathscr{S}(u')$ does not encounter the boundary of the near zone, and in this case ξ ranges from $\xi = -1$, at which $r' = \frac{1}{2}(ct - u' - r) = \frac{1}{2}(u - u')$, to $\xi = 1$, at which $r' = \frac{1}{2}(ct - u' + r) = \frac{1}{2}(u - u') + r$; these limits correspond to the events $\mathcal{A}$ and $\mathcal{B}$ in the left panel of Fig. 6.5. When $u - 2\mathcal{R} \leq u' \leq u$ we see that $\mathscr{S}(u')$ runs into the boundary of the near zone, and in this case the lower bound on r' must be $r' = \mathcal{R}$, with the corresponding value of $\xi > -1$ obtained from Eq. (6.102); the upper bound on r' is still $\frac{1}{2}(u - u') + r$, and these limits correspond to events $\mathcal{A}'$ and $\mathcal{B}'$ in the center panel of Fig. 6.5. The integration terminates when $u' = u$, as depicted on the right panel.

Defining $s := \frac{1}{2}(u - u')$ and the functions

$$A(s, r) := \int_{\mathcal{R}}^{r+s} \frac{P_\ell(\xi)}{r'^{(n-1)}}\, dr' , \tag{6.104a}$$

$$B(s, r) := \int_{s}^{r+s} \frac{P_\ell(\xi)}{r'^{(n-1)}}\, dr' , \tag{6.104b}$$

we obtain the final expression

$$\psi_{\mathscr{W}}(t, r, \theta, \phi) = \frac{n^{\langle L\rangle}}{r} \left\{ \int_0^{\mathcal{R}} ds\, f(\tau - 2s/c) A(s, r) + \int_{\mathcal{R}}^{\infty} ds\, f(\tau - 2s/c) B(s, r) \right\} \tag{6.105}$$

for the wave-zone contribution to the potential $\psi(x)$, when x is situated in the wave zone. The quantity ξ that appears in A and B is determined by Eq. (6.102), in which we insert the definitions $u = ct - r$ and $s = \frac{1}{2}(u - u')$; this yields

$$\xi = \frac{r + 2s}{r} - \frac{2s(r + s)}{rr'} , \tag{6.106}$$

with $\xi = 1$ when $r' = r + s$ and $\xi = -1$ when $r' = s$.

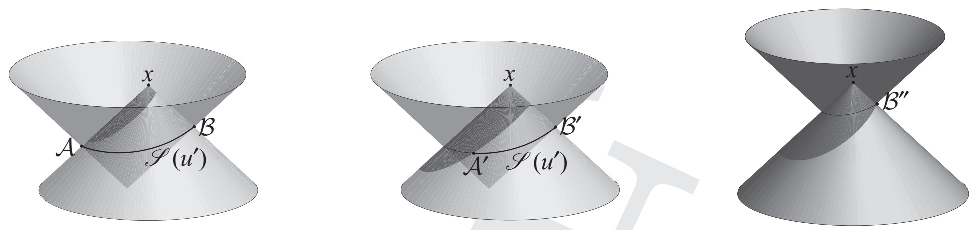

Fig. 6.6 Integration over the domain $\mathscr{W}(x)$, for a field point x in the near zone. The left panel corresponds to $u' < u - 2\mathcal{R}$; the integration runs from $\xi = -1$ (point $\mathcal{A}$) to $\xi = 1$ (point $\mathcal{B}$). The center panel corresponds to $u' > u - 2\mathcal{R}$; the intersection $\mathscr{S}(u')$ terminates at $\mathcal{A}'$, the boundary of the near zone $\mathscr{N}$. The right panel corresponds to $u' = u - 2\mathcal{R} + 2r$; the future cone intersects the past cone at $\xi = 1$ (point $\mathcal{B}''$) at the edge of the near zone.

Near-zone field point

We next take the field point x to be situated in the near zone, so that $r < \mathcal{R}$. In this case we find again that when $u' < u - 2\mathcal{R}$, $\mathscr{S}(u')$ does not encounter the near zone and ξ ranges from -1 to $+1$ (represented by the points $\mathcal{A}$ and $\mathcal{B}$ in the left panel of Fig. 6.6). When $u' > u - 2\mathcal{R}$, the integration runs from point $\mathcal{A}'$ in the center panel of Fig. 6.6, at which $r' = \mathcal{R}$, to point B', at which $\xi = 1$. But there is now a maximum value of u' at which the future null cone intersects $\mathscr{C}(x)$ at $\xi = 1$ (point $\mathcal{B}''$ in the right panel), corresponding to $u' = u - 2\mathcal{R} + 2r$; here the integration terminates. In this case, the minimum value of $s := \frac{1}{2}(u - u')$ is $\mathcal{R} - r$, and we obtain the expression

$$\psi_{\mathscr{W}}(t, r, \theta, \phi) = \frac{n^{\langle L\rangle}}{r}\left\{\int_{\mathcal{R}-r}^{\mathcal{R}} ds f(\tau - 2s/c)A(s, r) + \int_{\mathcal{R}}^{\infty} ds f(\tau - 2s/c)B(s, r)\right\} \tag{6.107}$$

for the wave-zone contribution to the potential $\psi(x)$, when x is situated in the near zone. The functions $A(s, r)$ and $B(s, r)$ are again given by Eq. (6.104), and ξ is still given by Eq. (6.106).

Equation (6.105) is a concrete expression for the $\psi_{\mathscr{W}}(x)$ of Eq. (6.93) when the field point x is in the wave zone, and Eq. (6.107) is the corresponding expression when x is in the near zone. In both cases the source function $\mu(x')$ takes the form displayed in Eq. (6.98), with $f(\tau')$ describing its temporal behavior, r'^{-n} describing its radial profile, and $n'^{\langle L\rangle}$ describing its angular profile. Note that $\psi_{\mathscr{W}}(x)$ depends on the entire past history of the system, because f must be evaluated at retarded times $\tau - 2s/c$ all the way back to $-\infty$. This is a direct consequence of the fact that the source μ is not bounded by the near zone, and is generated by retarded fields that are themselves solutions to the wave equation. In post-Minkowskian theory, this feature is a consequence of the non-linearity of the Einstein field equations, which imply that the gravitational field itself generates gravity. While it may seem like a daunting task to evaluate the integrals of Eqs. (6.105) and (6.107), we shall find that they can be evaluated relatively easily for many interesting situations, with physically reasonable assumptions about the past behavior to ensure convergence.

Estimates

It is instructive to give crude estimates to the integrals of Eqs. (6.105) and (6.107). Suppose first that we wish to evaluate Eq. (6.105) in the far-away wave zone, and keep only its dominant, r^{-1} part. Taking $P_\ell(\xi)$ to be of order unity, we approximate the functions defined by Eqs. (6.104) as $A \sim \int_{\mathcal{R}}^\infty p^{-(n-1)}\, dp \sim \mathcal{R}^{-(n-2)}$ and $B \sim \int_s^\infty p^{-(n-1)}\, dp \sim s^{-(n-2)}$; we ignore all numerical factors and exclude the special case $n = 2$. Inserting A into the first integral of Eq. (6.105) yields $\mathcal{R}^{-(n-2)} \int_0^{\mathcal{R}} f(\tau - 2s/c)\, ds$. Taking $\mathcal{R}$ to be small, we Taylor-expand $f(\tau - 2s/c)$ about $s = 0$ and integrate term by term. A typical term in the expansion is

$$\frac{\mathcal{R}^{q+1}}{c^q \mathcal{R}^{n-2}} f^{(q)}(\tau),$$

where the superscript (q) indicates the number of derivatives with respect to τ. As was motivated in the paragraph that follows Eq. (6.79), we are interested in the $\mathcal{R}$-independent part of $\psi_{\mathscr{W}}$. In order to extract this from our previous expansion, we retain the term $q = n - 3$ and discard all others. An estimate for the first integral is therefore $c^{-(n-3)} f^{(n-3)}(\tau)$. We next substitute B into the second integral of Eq. (6.105) and obtain $\int_{\mathcal{R}}^\infty s^{-(n-2)} f(\tau - 2s/c)\, ds$. Assuming that f and all its derivatives vanish in the infinite past, repeated integration by parts returns an expression of the form

$$\frac{f(\tau - 2\mathcal{R}/c)}{\mathcal{R}^{n-3}} + \frac{f^{(1)}(\tau - 2\mathcal{R}/c)}{c\mathcal{R}^{n-4}} + \frac{f^{(2)}(\tau - 2\mathcal{R}/c)}{c^2\mathcal{R}^{n-5}} + \cdots$$

The $\mathcal{R}$-independent part of this is easily seen to be of the form $c^{-(n-3)} f^{(n-3)}(\tau)$, as we had for the first integral. We conclude that a crude estimate for Eq. (6.105) is

$$\psi_{\mathscr{W}} \sim \frac{1}{c^{n-3}} \frac{n^{\langle L \rangle}}{r} f^{(n-3)}(\tau) \qquad \text{(far-away wave zone).} \tag{6.108}$$

The estimate ignores numerical factors, $\mathcal{R}$-dependent terms, and terms that decay faster than r^{-1}.

This estimate leads us to expect that the contribution from the wave-zone integral will be a small correction at any given iteration order of post-Minkowskian theory. First, the source function f is built from the pseudotensors $t_{\rm LL}^{\alpha\beta}$ and $t_{\rm H}^{\alpha\beta}$, which are quadratic in $h^{\alpha\beta}$ and therefore much smaller than the potentials themselves. Second, depending on n, the power with which the source falls off with r^{-1}, there will be additional time derivatives acting on f, generating additional powers of v_c/c. Accordingly, in many cases we will be able to ignore the contributions of the wave-zone integrals. But even when we are required to calculate those contributions, we will be able to do so using only the leading-order contributions to f. We will see a specific example of such a calculation in Sec. 7.4.

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Suppose next that we wish to evaluate Eq. (6.107) deep within the near zone, for $r \ll \mathcal{R}$. Here the first integral of Eq. (6.107) is approximated as $\int_{\mathcal{R}-r}^{\mathcal{R}} ds f(\tau - 2s/c)A(s, r) \sim r f(\tau - 2\mathcal{R}/c)A(\mathcal{R}, r)$, with $A(\mathcal{R}, r) \sim \int_{\mathcal{R}}^{r+\mathcal{R}} p^{-(n-1)}\, dp \sim r\mathcal{R}^{-(n-1)}$. This produces the estimate

$$\frac{r^2}{\mathcal{R}^{n-1}} f(\tau - 2\mathcal{R}/c)$$

for the first integral, and the $\mathcal{R}$-independent part of this is $c^{-(n-1)} r^2 f^{(n-1)}(\tau)$. The second integral of Eq. (6.107) involves the domain of integration $\mathcal{R} < s < \infty$. Because s is large compared with r, we have the estimate $B \sim \int_s^{r+s} p^{-(n-1)}\, dp \sim r s^{-(n-1)}$. Inserting this inside the integral gives $r \int_{\mathcal{R}}^{\infty} s^{-(n-1)} f(\tau - 2s/c)\, ds$, and repeated integration by parts returns an expression of the form

$$\frac{r f(\tau - 2\mathcal{R}/c)}{\mathcal{R}^{n-2}} + \frac{r f^{(1)}(\tau - 2\mathcal{R}/c)}{c\mathcal{R}^{n-3}} + \frac{r f^{(2)}(\tau - 2\mathcal{R}/c)}{c^2\mathcal{R}^{n-4}} + \cdots$$

The $\mathcal{R}$-independent part of this is of the form $c^{-(n-2)} r f^{(n-2)}(\tau)$. Collecting results, we conclude that a crude estimate for Eq. (6.107) is

$$\psi_{\mathscr{W}} \sim \frac{1}{c^{n-2}} n^{\langle L\rangle} \Big[f^{(n-2)}(\tau) + c r f^{(n-1)}(\tau) \Big] \qquad \text{(near zone).} \tag{6.109}$$

The estimate ignores numerical factors and all $\mathcal{R}$-dependent terms. In Sec. 7.3.4 we will learn that these contributions can be completely ignored for all our purposes in this book.

The case $n = 2$, for which μ falls off as r^{-2}, is special because the functions A and B are now logarithmic in $\mathcal{R}$ and s, and thus cannot be handled by our simple power-counting methods. We shall see that such terms are important in post-Minkowskian theory, and generate what are known as gravitational-wave "tails." We perform these computations, and describe these effects, in Chapter 11.

Box 6.7 **Solution to the wave equation**

The solution to the wave equation $\Box\psi = -4\pi\mu$ can be decomposed as

$$\psi = \psi_{\mathscr{N}} + \psi_{\mathscr{W}} ,$$

where $\psi_{\mathscr{N}}$ is the near-zone portion of the integral over the past light-cone $\mathscr{C}(x)$ of the field-point x, while $\psi_{\mathscr{W}}$ is the wave-zone portion. The boundary between the near and wave zones is placed at $r' = \mathcal{R} = O(\lambda_c)$, where λ_c is a characteristic wavelength of the radiation.

When the field point $x = (ct, \boldsymbol{x})$ is in the wave zone,

$$\psi_{\mathscr{N}}(x) = \sum_{\ell=0}^{\infty} \frac{(-1)^\ell}{\ell!} \partial_L \left[\frac{1}{r} \int_{\mathscr{M}} \mu(\tau, \boldsymbol{x}') x'^L \, d^3x' \right],$$

$$\psi_{\mathscr{W}}(x) = \frac{n^{\langle L\rangle}}{r} \left\{ \int_0^{\mathcal{R}} ds f(\tau - 2s/c)A(s, r) + \int_{\mathcal{R}}^{\infty} ds f(\tau - 2s/c)B(s, r) \right\}.$$

And when x is in the near zone,

$$\psi_{\mathcal{N}}(x) = \sum_{\ell=0}^{\infty} \frac{(-1)^\ell}{\ell! c^\ell} \left(\frac{\partial}{\partial t}\right)^\ell \int_{\mathscr{M}} \mu(t, \boldsymbol{x}') |\boldsymbol{x} - \boldsymbol{x}'|^{\ell-1} \, d^3x',$$

$$\psi_{\mathscr{W}}(x) = \frac{n^{\langle L \rangle}}{r} \left\{ \int_{\mathcal{R}-r}^{\mathcal{R}} ds f(\tau - 2s/c) A(s,r) + \int_{\mathcal{R}}^{\infty} ds f(\tau - 2s/c) B(s,r) \right\}.$$

Here $\tau := t - r/c$ is retarded time, $\mathscr{M}$ is a surface of constant time bounded externally by the sphere $r' = \mathcal{R}$, and L is a multi-index that contains a number ℓ of individual spatial indices. For $\psi_{\mathscr{W}}$ we have assumed that the source function μ is of the specific form

$$\mu(x) = \frac{1}{4\pi} \frac{f(\tau)}{r^n} n^{\langle L \rangle},$$

in which $\boldsymbol{n} = \boldsymbol{x}/r$, and we have defined

$$A(s,r) := \int_{\mathcal{R}}^{r+s} \frac{P_\ell(\xi)}{r'^{(n-1)}} \, dr', \qquad B(s,r) := \int_{s}^{r+s} \frac{P_\ell(\xi)}{r'^{(n-1)}} \, dr',$$

where $\xi = (r + 2s)/r - 2s(r+s)/(rr')$.

6.4 Bibliographical notes

The formulation of the Einstein field equations detailed in Sec. 6.1 was first proposed by Landau and Lifshitz in their classic textbook *The Classical Theory of Fields*, now available in a fourth English edition (2000). Rigorous definitions for the total mass, momentum, and angular momentum of an asymptotically-flat spacetime were provided in a sequence of papers by Arnowitt, Deser, and Misner; their work is based on Hamiltonian methods, and is conveniently summarized in their 1962 review article.

The relaxation of the Einstein field equations described in Sec. 6.2 has become a standard tool of the field. The idea originated in Havas and Goldberg (1962), and it is beautifully summarized in Ehlers *et al.* (1976); another useful reference is Walker and Will (1980). The curved-spacetime formulation of the relaxed field equations in Box 6.3 was first proposed by Thorne and Kovacs (1975).

The mathematical methods introduced in Sec. 6.3 to integrate the wave equation when the source is extended over all space were first devised by Wiseman and Will (1991). They form the core of the DIRE approach (Direct Integration of the Relaxed Einstein equations) to post-Minkowskian theory, initiated by Will and Wiseman (1996) and developed systematically by Pati and Will (2000 and 2001). An alternative approach, based on a formal multipolar expansion of the potential outside the source, was pursued by Blanchet, Damour, Iyer, and their collaborators; this work is nicely summarized in Blanchet's *Living Reviews* article (2006).

6.5 Exercises

6.1 Show that $g_{\alpha\beta} = \sqrt{-\mathfrak{g}}\,\mathfrak{g}_{\alpha\beta}$, where $\mathfrak{g}_{\alpha\beta}$ is the matrix inverse to $\mathfrak{g}^{\alpha\beta}$, and $\mathfrak{g} = \det[\mathfrak{g}^{\alpha\beta}] = g$. If we define $\mathfrak{g}^{\alpha\beta} := \eta^{\alpha\beta} - h^{\alpha\beta}$, and $h^{\alpha\beta}$ is of order G, show that

$$(-g) = 1 - h + \frac{1}{2}h^2 - \frac{1}{2}h^{\mu\nu}h_{\mu\nu} + O(G^3)\,,$$

$$g_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta} - \frac{1}{2}h\eta_{\alpha\beta} + h_{\alpha\mu}h^{\mu}{}_{\beta} - \frac{1}{2}hh_{\alpha\beta} + \left(\frac{1}{8}h^2 - \frac{1}{4}h^{\mu\nu}h_{\mu\nu}\right)\eta_{\alpha\beta} + O(G^3)\,,$$

where indices on $h^{\alpha\beta}$ are lowered and contracted with the Minkowski metric.

6.2 Show that under the coordinate transformation $x'^{\mu} = f^{\mu}(x^{\alpha})$,

$$\mathfrak{g}^{\mu'\nu'} = J^{-1}\partial_\alpha f^\mu \partial_\beta f^\nu \mathfrak{g}^{\alpha\beta}\,,$$

$$\partial_{\nu'}\mathfrak{g}^{\mu'\nu'} = \sqrt{-g'}\,\Box_g f^\mu(x^\alpha)\,,$$

where $J := \det[\partial f^\mu/\partial x^\alpha]$ is the Jacobian of the transformation, and where for any scalar function f, $\Box_g f = (-g)^{-1/2}\partial_\beta(\mathfrak{g}^{\alpha\beta}\partial_\alpha f)$.

6.3 Consider the Schwarzschild metric in harmonic coordinates, given by Eqs. (5.171). Show explicitly that

$$\mathfrak{g}^{00} = -\frac{(1+R/2r)^3}{1-R/2r}\,,$$

$$\mathfrak{g}^{jk} = \delta^{jk} - \left(\frac{R}{2r}\right)^2 n^j n^k\,,$$

where $R := 2GM/c^2$, and verify that the harmonic gauge condition $\partial_\beta\mathfrak{g}^{\alpha\beta} = 0$ is satisfied.

6.4 Consider the potentials $h^{\alpha\beta}$ for a stationary source ($\partial_0 h^{\alpha\beta} = 0$), in harmonic gauge. Show that the conserved quantities for the spacetime can be written in terms of the following surface integrals at infinity:

$$M = -\frac{c^2}{16\pi G}\oint_\infty r^2\frac{\partial h^{00}}{\partial r}d\Omega\,,$$

$$P^j = -\frac{c^3}{16\pi G}\oint_\infty r^2\frac{\partial h^{0j}}{\partial r}d\Omega\,,$$

$$J^{jk} = -\frac{c^3}{16\pi G}\oint_\infty r^2\frac{\partial}{\partial r}\left(x^j h^{0k} - x^k h^{0j}\right)d\Omega\,,$$

$$R^j = -\frac{c^2}{16\pi GM}\oint_\infty r^4\frac{\partial}{\partial r}\left(\frac{x^j h^{00}}{r^2}\right)d\Omega\,,$$

where $d\Omega = \sin\theta\, d\theta d\phi$ is the element of solid angle.

6.5 Consider the stationary metric given by

$$ds^2 = -\left(1 - \frac{R}{r}\right) d(ct)^2 - \frac{4GS}{c^2 r} \sin^2\theta \, d\phi dt + \left(1 + \frac{R}{r}\right)\left(dr^2 + r^2\, d\theta^2 + r^2 \sin^2\theta \, d\phi^2\right),$$

which is accurate to first order in G in a post-Minkowskian expansion; here $R = 2GM/c^2$ and S is a constant.

(a) Working to first order in G, find $\mathfrak{g}^{\alpha\beta}$ and verify that it is in the harmonic gauge.
(b) Using the surface integral formulation, find the mass, momentum, and angular momentum for this spacetime.

6.6 Using surface integrals, find the center-of-mass position of a spacetime for which

$$h^{00} = \frac{4GM}{c^2|\boldsymbol{x} - \boldsymbol{a}|},$$

where $\boldsymbol{a}$ is a constant vector.

6.7 Verify that the harmonic energy-momentum pseudotensor is conserved, so that $\partial_\beta[(-g)t_{\rm H}^{\alpha\beta}] = 0$.

6.8 Using the techniques of Sec. 6.3, find the solution to the wave equation $\Box\psi = -4\pi\mu$ when $\mu = -\boldsymbol{p}\cdot\boldsymbol{\nabla}\delta(\boldsymbol{x})\cos\omega t$, with $\boldsymbol{p}$ a constant vector. First take x to be in the wave zone, and find the solution there; then take x to be in the near zone. For the wave-zone expression, show that the sum over ℓ truncates. For the near-zone expression, show that the sum does not truncate. Compare your results with those of Box 6.6. Can you reconcile your results with the exact solution?

6.9 Using the techniques of Sec. 6.3, find the solution to the wave equation $\Box\psi = -4\pi\mu$ when μ is equal to $\mu_0(r/r_0)^4$ for $r < r_0$, and to $\mu_0(r_0/r)^4$ for $r > r_0$. You may take r_0 to be smaller than $\mathcal{R}$. You should find that

$$\psi = 4\pi\mu_0 r_0^2\left[\frac{2}{3} - \frac{1}{42}\left(\frac{r}{r_0}\right)^6\right]$$

for $r < r_0$, and

$$\psi = 4\pi\mu_0 \frac{r_0^3}{r}\left(\frac{8}{7} - \frac{r_0}{2r}\right)$$

when $r > r_0$. Observe that while $\psi_{\mathcal{N}}$ and $\psi_{\mathcal{W}}$ both depend on $\mathcal{R}$, the final outcome for ψ is independent of $\mathcal{R}$.